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ABSTRACT

This is part two of a two-part manual for teachers using SMSG high school text materials. A chapter-by-chapter commentary on the text and answers to all the exercises are given. Chapter topics include: (1) area and integral; (2) basic integral theorems; (3) logarithmic and exponential functions; (4) growth, decay and competition; and (5) integration. (MP)

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CALCULUS

PART II

SE 028 241

SCHOOL MATHEMATICS STUDY GROUP

Calculus

Part 2 Teacher's Commentary

REVISED EDITION

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Teacher's Commentary

Chapter 6

AREA AND INTEGRAL

In Section 6-1 we use the word "region" without definition. For the purposes of this text a region is the interior of a piecewise smooth simple closed curve plus the boundary curve. Here the aim of the section is merely to give an intuitive introduction to the concept of area. Thus, we make no attempt to examine subtleties like the proof that the boundary of a region has zero area.

The fundamental existence theory of the integral is covered in Sections 6-3 and A6-2. The basic existence Theorem 6-3a, can be proved by very simple means, but the proof requires the completeness of the real number system, Section A1-5, and therefore is left to the appendix. The Separation Axiom is a form of completeness postulate which is most appropriate for the proof and permits an elegant exposition. If it has not been done before, and if a discussion of completeness is appropriate for your syllabus, then the discussion of Section 6-3(i) represents an excellent introduction to the concept. A suggested organization of the material with completeness incorporated is the following: Sections 6-1, 6-2, 6-3 (to Definition 6-3), A1-5 (at least through Lemma A1-5), Section 6-3(from Definition 6-3 to the end of Section 6-3(ii)), A6-2 (at least through Theorem 6-3; for the slight extra effort it is also worth while to do Lemma A6-2c), Section 6-3(iii) to the end of the chapter.

Note that Lemma A6-2c is used in the proof of Theorem 6-4b.

The remainder of A6-2, from Lemma A6-2d on, is devoted to a proof that the concept of integral defined in the text (the Darboux integral) is equivalent to that of Riemann integral. Appendix 7 proves the integrability of continuous functions. These appendices are aimed at the more sophisticated student.

In Section 6-5(ii), the Fundamental Theorem of Calculus is anticipated. The principal result, Equation (3), is not singled out as a theorem because it will be convenient to restrict the integrand to be continuous in order to obtain the converse, that the derivative of the integral is the integrand.

Solutions Exercises 6-1

1. Read Section 1-2 carefully and locate the places in the discussion where the four properties of area are used.

In Figure 1-2a, the area of the standard region under the graph of f_2 on $[a,b]$ is the sum of the area of the shaded region plus that of the standard region under the graph of f_1 on $[a,b]$ (Property 3). In Figure 1-2d, the area A of the standard region under the graph of f satisfies

$$m(b - a) \leq A \leq M(b - a) .$$

(Properties 2 and 4).

For Figure 1-2e, the relation

$$m(x_2 - x_1) \leq m^*(x_2 - x_1) \leq A^* \leq M^*(x_2 - x_1) \leq M(x_2 - x_1)$$

holds (Properties 2 and 4).

When we form the totals of the estimates from above and below for the subregions to estimate the area of the entire region we are using Properties 2 and 3.

2. Prove from Property 3 that if a region R is the union of n non-overlapping regions then

$$\alpha(R) = \alpha(R_1) + \alpha(R_2) + \dots + \alpha(R_n) .$$

We have

$$\begin{aligned}\alpha(R_1 \cup R_2 \cup R_3 \cup \dots \cup R_n) &= \alpha(R_1) + \alpha(R_2 \cup R_3 \cup \dots \cup R_n) \\ &= \alpha(R_1) + \alpha(R_2) + \alpha(R_3 \cup \dots \cup R_n) \\ &= \dots \\ &= \alpha(R_1) + \alpha(R_2) + \alpha(R_3) + \dots + \alpha(R_n) .\end{aligned}$$

The argument may be formalized by the use of mathematical induction.

3. Show that Property 2 is actually a consequence of Property 3 given that area is nonnegative.

Let S be a subregion of T and let R be the complementary region of S in T; i.e., R is the region which does not overlap S and for which

$R \cup S = T$. (We deliberately omit the question of existence of R .) Since $\alpha(R) \geq 0$ and

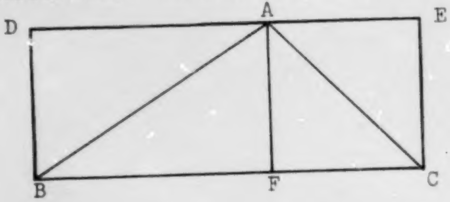
$$\alpha(R) + \alpha(S) = \alpha(T)$$

we have the desired result

$$\alpha(S) = \alpha(T) - \alpha(R) \leq \alpha(T) .$$

4. (a) Using the given properties of area obtain the area of a triangle by elementary geometrical arguments.

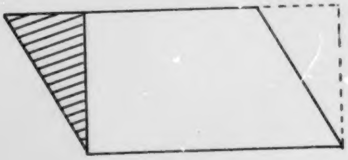
Let $\triangle ABC$ be the triangle and let BC be the longest side. We inscribe $\triangle ABC$ in a rectangle with one side on BC and suppose the foot of the perpendicular from A to BC lies on BC at F . (See Figure.)



From elementary geometry we have $\triangle AFC$ congruent to $\triangle CEA$, and $\triangle AFB$ to $\triangle BDA$. It follows that the area of the triangle ABC is half the area of the rectangle $BCED$ and hence equal to half the product of the base BC and the altitude AF .

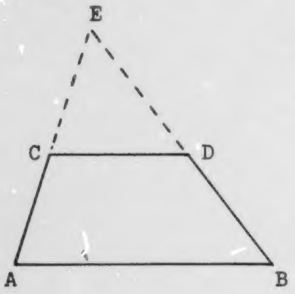
In this "proof" it is assumed from elementary geometry that congruent regions have equal area. It is possible to deduce this result from an analytical definition of area (see Section 11-6).

(b) Do the same for a trapezoid.



(i) If the sides of the trapezoid are parallel, the trapezoid is a parallelogram and has the same area as a rectangle of the same base and altitude.

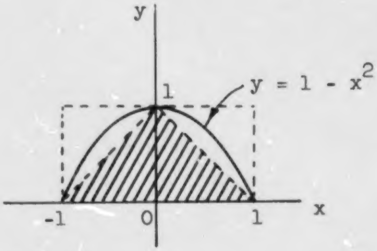
(ii) If the sides are not parallel, extend them until they meet. The area is then the difference between the area of $\triangle EAB$ and $\triangle ECD$. In either case we get the usual formula.



5. Estimate the area of each region described below.

(a) $\{(x,y) : y \geq 0\} \cap \{(x,y) : y \leq 1 - x^2\}$

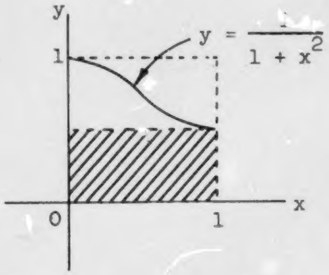
$1 \leq A \leq 2$ or any such validly obtained estimate (actual value, $\frac{4}{3}$).



(b) $\{(x,y) : 0 \leq y \frac{1}{1 + x^2}\} \cap \{(x,y) : 0 \leq x \leq 1\}$

$\frac{1}{2} \leq A \leq 1$

(actual value, $\frac{\pi}{4} = .785 \ldots$).



6. If Property 4 is replaced by

Property 4^* : The area of a unit square is one,
Property 5 : Congruent regions have the same area,

show that the area of a square whose side is of length a is a^2.

The proof is given first for rational a then for arbitrary real a. If $a = \frac{1}{n}$ for a natural number n, then from the observation that the unit square can be subdivided into n^2 squares of sidelength $\frac{1}{n}$ it follows from Property 3 and new Properties 4^* and 5 that the area of the square is $\frac{1}{n^2}$. If $a = \frac{m}{n}$, then the square may be subdivided into m^2 congruent squares of sidelength $\frac{1}{n}$ and from the preceding result, the area is $\frac{m^2}{n^2}$. If now a is any real number, take $m = [\![an]\!]$. Then $m \leq an < m + 1$. It follows that the given square contains a square of sidelength $\frac{m}{n}$ and is contained in a square of sidelength $\frac{m+1}{n}$. Consequently, for the area A of the given square, by Property 2,

$$\frac{m^2}{n^2} \leq A < \frac{(m+1)^2}{n^2} .$$

Consequently, for all natural numbers n,

$$\frac{(an-1)^2}{n^2} < A < \frac{(an+1)^2}{n^2} ,$$

or

$$a^2 - \frac{2a}{n} + \frac{1}{n^2} < A < a^2 + \frac{2a}{n} + \frac{1}{n^2} ;$$

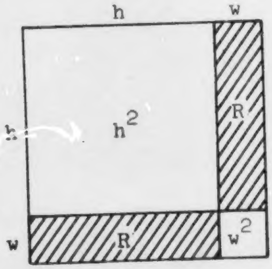
hence

$$|A - a^2| < \frac{2a+1}{n} ,$$

from which the result follows.

7. Using Number 6, show that the area of a rectangle of height h and width w is hw.

Form the square of side $h + w$. If A is the area of the given rectangle, then from the figure and from Number 6 $(h+w)^2 = 2R + h^2 + w^2$. It follows that $R = hw$.



Solutions Exercises 6-2

1. (Requires Section A3-2(ii) for parts (d) and (e).) Use the summation method to find the area of the standard region defined by
(a) $f : x \longrightarrow c \;, 0 \leq x \leq b \;, c > 0$.

Use a subdivision of the interval into n equal parts. Define $\alpha(S)$, $\alpha(T)$, and $\alpha(R)$ as in the text, using the respective minimum and maximum values of f in each subinterval. In this case the maximum on any interval is equal to the minimum, so that

$$\alpha(S) = \alpha(T) = \sum_{k=1}^{n} c \cdot \frac{b}{n} = bc \;.$$

(b) $f : x \longrightarrow cx \;, 0 \leq x \leq b \;, c > 0$.

Here

$$\alpha(S) = \sum_{k=1}^{n} c \cdot \frac{(k-1)b}{n} \cdot \frac{b}{n} = \frac{cb^2}{n^2} \sum_{k=1}^{n} (k-1)$$

$$= \frac{cb^2}{n^2} \frac{n(n-1)}{2} \;,$$

and

$$\alpha(T) = \sum_{k=1}^{n} c \cdot \frac{kb}{n} \cdot \frac{b}{n} = \frac{cb^2}{n^2} \sum_{k=1}^{n} k = \frac{cb^2}{n^2} \frac{n(n+1)}{2} \;.$$

From $\alpha(S) \leq \alpha(R) \leq \alpha(T)$ it follows that

$$-\frac{cb^2}{2n} \leq \alpha(R) - \frac{cb^2}{2} \leq \frac{cb^2}{2n} \;,$$

for each natural number n . Consequently,

$$\alpha(R) = \frac{cb^2}{2} \;.$$

(c) $f : x \longrightarrow x^2 + 2x$, $0 \leq x \leq b$.

Here

$$\alpha(T) = \sum_{k=1}^{n}\left[\left(\frac{kb}{n}\right)^2 + 2\,\frac{kb}{n}\right]\frac{b}{n}$$

$$= \frac{b^3}{n^3}\sum_{k=1}^{n} k^2 + \frac{2b^2}{n^2}\sum_{k=1}^{n} k$$

$$= \frac{b^3}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) + \frac{2b^2}{n^2}\,\frac{n(n + 1)}{2}$$

$$= \frac{b^3}{3} + b^2 + \frac{b^2}{n} + \frac{b^3}{2n} + \frac{b^3}{6n^2}$$

and

$$\alpha(S) = \alpha(T) - \frac{b^3}{n} - \frac{2b^2}{n} .$$

As in Part (b), from $\alpha(S) \leq \alpha(R) \leq \alpha(T)$ obtain

$$\alpha(R) = \frac{b^3}{3} + b^2 .$$

(d) $f : x \longrightarrow \sin(ax + b)$, $0 \leq x \leq c$; a , b , c such that $\sin(ax + b) \geq 0$ on $[0,c]$.

The interval $[0,c]$ may be subdivided into two subintervals where f is strongly monotone. For simplicity, assume f is increasing on $[0,c]$. Then

$$\alpha(T) = \sum_{k=1}^{n} \frac{c}{n} \sin\left(\frac{akc}{n} + b\right) .$$

In A3-2(ii), Equation (6), take $\frac{ac}{n}$ for a and $b + \frac{ac}{2n} - \frac{\pi}{2}$ for b .

Then

$$\alpha(T) = \frac{c}{n}\sum_{k=1}^{n} \sin(\frac{akc}{n} + b) = \frac{c \sin(\frac{ac}{2} + b + \frac{ac}{2n})\sin \frac{ac}{2}}{n \sin \frac{ac}{2n}} .$$

Consequently, from the continuity of cos and $\lim_{x \sim 0} \frac{\sin x}{x}$,

$$\lim_{n \sim \infty} \alpha(T) = \frac{2}{a} \sin(\frac{ac}{2} + b)\sin \frac{ac}{2} = \frac{\cos(ac + b) - \cos b}{a} .$$

A similar argument yields the same limit for $\alpha(S)$. By the Squeeze Theorem, it follows that $\alpha(R)$ is this common limit.

(e) $f : x \longrightarrow \cos^2 x$, $0 \leq x \leq c$.
Use $\cos^2 x = \frac{\cos 2x + 1}{2}$. Proceed as in Part (d) with
$\cos 2x = \sin(2x + \frac{\pi}{2})$.

$$\alpha(R) = \frac{\sin 2c}{4} + \frac{c}{2} .$$

2. The problem posed in Section 1-2 was to determine the area of the standard region $f : x \longrightarrow \sqrt{x}$ on $[0,1]$. The summation encountered there was similar to the one encountered in this section. Use this fact to solve the problem of Section 1-2.

The upper sum of Section 1-2 has the form

$$\begin{aligned}\alpha(T) &= \frac{1}{n^3}\sum_{k=1}^{n} k(2k - 1)\\ &= \frac{1}{n^3}\left[2\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k\right]\\ &= \frac{1}{n^3}\left[\frac{2n^3}{3} + n^2 + \frac{n}{3} - \frac{n(n + 1)}{2}\right]\\ &= \frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2} .\end{aligned}$$

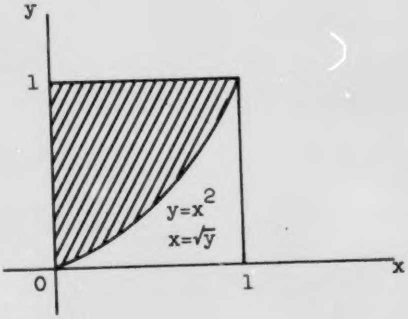
We have also

$$\alpha(S) = \alpha(T) - \frac{1}{n} = \frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2} .$$

It follows that the area is $\frac{2}{3}$.

3. Obtain the result of Exercise 2 using only the fact that the area under $f : x \longrightarrow x^2$ on $[0,1]$ is $\frac{1}{3}$, together with the basic properties of area, without resort to summation techniques.

By Property 3, the area of the standard region under the graph of $x = \sqrt{y}$ on $[0,1]$ (the shaded region) plus the area of the standard region under the graph $y = x^2$ on $[0,1]$ (unshaded region) is 1 .

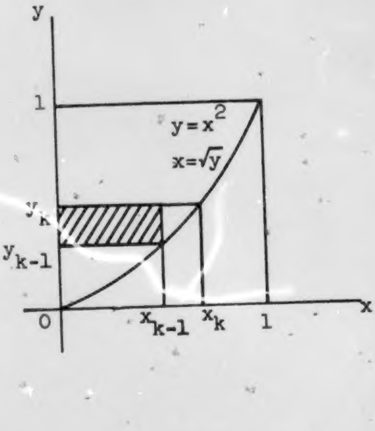


4. Show how the upper estimating sums for $\sqrt{x}$ are related term-by-term to the lower estimating sums for x^2 . (Hint: Sketch a graph of $y = x^2$. Use this graph and the y-axis to represent the standard region defined by $\sqrt{x}$.)

Observe from the figure that a term from the upper sum (area of the unshaded rectangle) for $f : x \longrightarrow x^2$ of the form $x_k^2(x_k - x_{k-1})$, corresponds to a term of the lower sum (area of the shaded rectangle) for $g : y \longrightarrow \sqrt{y}$ of the form

$$\sqrt{y_{k-1}}\,(y_k - y_{k-1})$$

where $y_k = x_k^2$. Furthermore, the sum of the two is $x_k y_k - x_{k-1}y_{k-1}$. Adding the upper sum for f to the lower sum for g , we have



$$(x_1y_1 - x_0y_0) + (x_2y_2 - x_1y_1) + \dots + (x_ny_n - x_{n-1}y_{n-1})$$
$$= x_ny_n - x_0y_0$$
$$= 1 .$$

5. If $S_n = \sqrt{1} + \sqrt{2} + \dots + \sqrt{n}$, show that

$$\frac{2}{3}\sqrt{n^3} < S_n < \frac{2}{3}\sqrt{n^3} + \sqrt{n} .$$

Divide $[0,1]$ into n equal subintervals. An upper bound for the area $\alpha(R)$ of the standard region $f : x \longrightarrow \sqrt{x}$ on $[0,1]$ is

$$\alpha(T) = \sum_{k=1}^{n} \sqrt{\frac{k}{n}} \cdot \frac{1}{n} = \frac{1}{\sqrt{n^3}} S_n ;$$

a lower bound is

$$\alpha(S) = \sum_{k=1}^{n} \sqrt{\frac{k-1}{n}} \cdot \frac{1}{n} = \frac{1}{\sqrt{n^3}} (S_n - \sqrt{n}) .$$

Since

$$\alpha(S) \leq \alpha(R) \leq \alpha(T)$$

by Property 2, and $\alpha(R) = \frac{2}{3}$ by Number 2, the result follows immediately.

Solutions Exercises 6-3

1. Evaluate the integral of each function f over the indicated interval.

   (a) $f(x) = 2 - x^2$ $\qquad 0 \le x \le 1$;

   (b) $f(x) = x$, $\qquad 1 \le x \le 2.5$;

   (c) $f(x) = \frac{5}{2}$, $\qquad 2.5 \le x \le 3$;

   (d) $f(x) = 5 - x$, $\qquad 3 \le x \le 5$.

In each of the following we use subdivision into n equal parts.

(a) Since f is monotone decreasing, we take

$$L = \sum_{k=1}^{n} (2 - x_k^2)(x_k - x_{k-1})$$

$$= \sum_{k=1}^{n} \left(2 - \frac{k^2}{n^2}\right)\frac{1}{n} \qquad \left(x_k = \frac{k}{n}\right)$$

$$= \frac{1}{n}\sum_{k=1}^{n} 2 - \frac{1}{n^3}\sum_{k=1}^{n} k^2$$

$$= 2 - \frac{1}{3} - \frac{1}{2n} - \frac{1}{6n^2} .$$

Since $U = L + \frac{1}{n} - \frac{5}{3} + \frac{1}{2n} - \frac{1}{6n^2}$, the integral is $\frac{5}{3}$.

(b) Take

$$U = \sum_{k=1}^{n} (1 + kh)h , \qquad (x_k = 1 + kh)$$

$$L = U(\sigma) - 1.5h$$

where

$$h = \frac{(2.5 - 1)}{n} = \frac{1.5}{n}$$

$$U = \sum_{k=1}^{n} h + h^2 \sum_{k=1}^{n} k$$

$$= nh + h^2 \frac{n(n + 1)}{2} .$$

Then, since $nh = 1.5$,

$$U = 1.5 + \frac{(1.5)^2(1 + \frac{1}{n})}{2} .$$

It follows that the integral is

$$1.5 + \frac{2.25}{2} = 2.625 .$$

(c) We take for both upper and lower sums

$$U = L = \sum_{k=1}^{n} \frac{5}{2} h$$

where $h = \frac{(3 - 2.5)}{n} = \frac{0.5}{n}$.

$$\sum_{k=1}^{n} \frac{5}{2} h = \frac{5nh}{2} = \frac{5}{2}(0.5) = 1.25 .$$

(d) We take for the lower sum

$$L = \sum_{k=1}^{n} [5 - (3 + kh)]h$$

and take $U = L + 2h$ where $h = \frac{(5 - 3)}{n} = \frac{2}{n}$.

$$L = h \sum_{k=1}^{n} (2 - kh)$$

$$= h \sum_{k=1}^{n} 2 - h^2 \sum_{k=1}^{n} k$$

$$= 2nh - \frac{h^2 n(n + 1)}{2}$$

$$= 4 - \frac{4(1 + \frac{1}{n})}{2}$$

$$= 2 - \frac{2}{n}$$

It follows that the integral is 2 .

2. (a) Find the minimum and the maximum values of $f(x) = 2 + 2x - x^2$ on the interval $[0,1]$, and use them to find two numbers respectively below and above the value of $\int_0^1 f(x)\, dx$.

$f'(x) = 2 - 2x$ is zero when $x = 1$.

$f(0) = 2$ and $f(1) = 3$.

Max: $f(x) = 3$, Min: $f(x) = 2$ for x in $[0,1]$.

Hence $U = 3 \cdot 1 = 3$ and $L = 2 \cdot 1 = 2$.

(b) Check your result by evaluating the integral.

Use the summations of Exercises 6-2, Number 1 and observe that upper and lower sums for x^2 are the negatives of lower and upper sums, respectively, for $-x^2$.

$$\int_0^1 f(x)\,dx = 2 + 1 - \frac{1}{3} = 3 - \frac{1}{3} ,$$

a value between 2 and 3.

3. Find upper and lower sums differing by less than .1 for the area under $f(x) = \frac{1}{x}$ on the interval $[1,2]$.

Take a subdivision of $[1,2]$ into n equal subintervals and use the maximum and minimum of $\frac{1}{x}$ as bounds in the subinterval, we have

$$U - L = \left(1 - \frac{1}{2}\right)\frac{1}{n} = \frac{1}{2n} .$$

It is sufficient to take $\frac{1}{2n} < \frac{1}{10}$ or $n > 5$.
Taking $n = 6$ we obtain

$$U = \left(1 + \frac{6}{7} + \frac{6}{8} + \frac{6}{9} + \frac{6}{10} + \frac{6}{11}\right) \cdot \frac{1}{6} = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots + \frac{1}{11} ,$$

$$L = \left(\frac{6}{7} + \frac{6}{8} + \frac{6}{9} + \frac{6}{10} + \frac{6}{11} + \frac{6}{12}\right) \cdot \frac{1}{6} = \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots + \frac{1}{12} .$$

(A decimal representation of the answer is not required.)

4. Evaluate each of the following integrals.

(a) $\int_{-1}^{1} x^3\,dx$

(b) $\int_{-2}^{2} |x|\,dx$

(c) $\int_{-1}^{1} x^2\,dx$ (See Exercises 6-4, No. 4)

In each of the following use a subdivision into $2n$ equal parts, and set $h = \frac{(b - a)}{2n}$ where a and b are the lower and upper ends of integration respectively, and separate the sums for positive and negative values of x.

(a) Separate the upper sums into two sums over the intervals $[-1,0]$ and $[0,1]$.

$$U = \sum_{k=1}^{n} (kh)^3 h - \sum_{i=1}^{n} [(i - 1)^3 h^3]h$$

$$= \sum_{k=1}^{n} (kh)^3 h - \sum_{k=0}^{n-1} (kh)^3 h$$

$$= (nh)^3 h$$

$$= h \qquad \left(\text{from } h = \frac{2}{2n} = \frac{1}{n}\right).$$

Then

$$L = U - 2h = -h .$$

The integral is zero.

(b) Separate the upper sums into sums over the intervals $[-2,0]$, $[0,2]$.

$$U = \sum_{k=1}^{n} |kh|h + \sum_{k=1}^{n} |-kh|h$$

$$= 2 \sum_{k=1}^{n} (kh)h$$

$$= 2h^2 \sum_{k=1}^{n} k$$

$$= h^2 n(n + 1) \qquad (\text{where } hn = 2)$$

$$= 4 + 2h$$

and

$$L = U - 2h - 2h = 4 - 2h .$$

The value of the integral is 4.

(c) By the method of the preceding exercise we find

$$U = 2\sum_{k=1}^{n} (kh)^2 h = 2h^3 \sum_{k=1}^{n} k^2$$

and since $h = \frac{1}{n}$, from Section 6-2, we have

$$U = \frac{2}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)$$
$$= \frac{2}{3} + \frac{1}{n} + \frac{1}{3n^2}$$

$$L = U - 2h = U - \frac{2}{n}.$$

The integral is $\frac{2}{3}$.

5. Approximate $\int_0^1 \frac{1}{1+x^2}\,dx$ by Riemann sums.

Given a subdivision $\sigma = \{x_0, x_1, \ldots, x_n\}$ of $[0,1]$ we have

$$R = \sum_{k=1}^{n} \frac{1}{1+\xi_k^2}(x_k - x_{k-1})$$

where $x_{k-1} \leq \xi_k \leq x_k$. If we choose an equal subdivision and take $\xi_k = x_k$ we obtain the following approximations:

$$n = 1 \qquad \frac{1}{1+1}\cdot 1 = \frac{1}{2} = 0.50$$

$$n = 2 \qquad \left(\frac{1}{1+\frac{1}{4}} + \frac{1}{1+1}\right)\frac{1}{2} = 0.65$$

$$n = 3 \qquad \left(\frac{1}{1+\frac{1}{9}} + \frac{1}{1+\frac{4}{9}} + \frac{1}{1+1}\right)\frac{1}{3} \approx 0.70$$

(Exact value is $\frac{\pi}{4} = 0.785 \ldots$).

6. A function f defined on the interval $[a,b]$ is said to be a step-function on $[a,b]$ if for some partition $\sigma = \{x_0,x_1,\ldots,x_n\}$ of the interval, $f(x)$ is constant on each open subinterval (x_{k-1},x_k) , $k = 1$, 2 , ... , n . Thus $\operatorname{sgn} x$ is a step function on $[-1,1]$, where $\operatorname{sgn} x$ is defined by

$$\operatorname{sgn} x = \begin{cases} -1 , & x < 0 \\ 0 , & x = 0 \\ 1 , & x > 0 . \end{cases}$$

(a) Prove that a step-function is integrable.

Let c_k be the constant value of the step-function f in the subinterval (x_{k-1},x_k) . The maximum and minimum values of f on $[a,b]$ are given by

$$M = \max \{f(x_k) , c_k : k = 0 , 1 , \ldots , n\}$$

$$m = \min \{f(x_k) , c_k : k = 0 , 1 , \ldots , n\} .$$

Choose a partition $\tau = \{u_0,u_1,\ldots,u_{2n+1}\}$ where each point x_k $(k = 1,\ldots,n - 1)$ is the midpoint of the interval $[u_{2k},u_{2k+1}]$ and take

$$u_{2k+1} - x_k = x_k - u_{2k} = \epsilon < \nu(\sigma)$$
$$(k = 1 , \ldots , n - 1)$$

and

$$u_1 - x_0 = x_n - u_{2n} = \epsilon ;$$

that is, enclose the points of σ in intervals of small total length. For upper and lower sums take

$$U = \sum_{k=1}^{n} c_k(u_{2k} - u_{2k-1}) + \sum_{k=1}^{n-1} M(u_{2k+1} - u_{2k})$$
$$+ M(u_1 - u_0) + M(u_n - u_{n-1})$$

$$= \sum_{k=1}^{n} c_k(u_{2k} - u_{2k-1}) + 2\epsilon n M .$$

Similarly, we have

$$L = \sum_{k=1}^{n} c_k(u_{2k} - u_{2k-1}) + 2\epsilon n\, m\ .$$

It follows that,

$$U - L = 2\epsilon n(M - m)\ ,$$

and since ϵ can be taken arbitrarily small, that the upper and lower sums can be brought closer together than any fixed tolerance.

(b) Find $\int_a^b \operatorname{sgn} x\, dx$.

If $[a,b]$ does not include the origin there are two cases.

(i) $a > 0$. Since $\operatorname{sgn} x = 1$ on $[a,b]$, the integral has the value $b - a$.

(ii) $b < 0$. Since $\operatorname{sgn} x = -1$ on $[a,b]$ the integral has the value $-(b - a)$.

(iii) If $[a,b]$ includes the origin, $a \leq 0 \leq b$. We take a subdivision and upper and lower sums and obtain the integral I for each case as follows.

$a = 0$, $a = \{0,\epsilon,b\}$,

$$U = b\ ,\quad L = b - \epsilon$$
$$I = b$$

$a < 0 < b$, $\sigma = \{a,-\epsilon,\epsilon,b\}$, where $0 < \epsilon < \min\{-a,b\}$,

$$U = b + a + 2\epsilon$$
$$L = b + a - 2\epsilon$$
$$I = b + a$$

$b = 0$, $\sigma = \{a,-\epsilon,0\}$

$$U = a + \epsilon\ ,\quad L = a$$
$$I = a$$

These results can be summarized by the simple formula

$$\int_a^b \operatorname{sgn} x\, dx = |b| - |a|\ .$$

7. Evaluate each of the following integrals:

(a) $\int_{-1}^{3} [\![3x + 4]\!]\,dx$.   (c) $\int_{1}^{5} \sqrt{2[\![x]\!]}\,dx$.

(b) $\int_{0}^{10} [\![\frac{x}{4}]\!]\,dx$.   (d) $\int_{1}^{5} [\![\sqrt{2x}]\!]\,dx$.

Each of the given integrands is an increasing step-function and hence is integrable either by the monotone property or by Number 6. In the notation of the solution of Number 6a, the integral of a step-function is

$$\sum_{k=1}^{n} c_k(x_k - x_{k-1})$$

as can be deduced directly from the given upper and lower estimates. Apply this result as follows.

(a) $\int_{-1}^{3} [\![3x + 4]\!]dx = (1 + 2 + 3 + \dots + 12)\cdot\frac{1}{3} = 26$.

(b) $\int_{0}^{10} [\![\frac{x}{4}]\!]dx = (0 + 1)\cdot 4 + 2\cdot 2 = 8$.

(c) $\int_{1}^{5} \sqrt{2[\![x]\!]}\,dx = (\sqrt{2} + \sqrt{4} + \sqrt{6} + \sqrt{8})\cdot 1$

$= 2 + 3\sqrt{2} + \sqrt{6}$.

(d) $\int_{1}^{5} [\![\sqrt{2x}]\!]dx = 1\cdot 1 + 2\cdot\frac{5}{2} + 3\cdot\frac{1}{2} = \frac{15}{2}$.

8. Let $\sigma = \{x_0, x_1, x_2, \dots, x_n\}$ be a partition of $[a,b]$, and f , a function which is integrable over each interval $[x_{k-1}, x_k]$, $k = 1$, 2 , ... , n . Prove that f is integrable over the entire interval $[a,b]$ and that

$$\int_{a}^{b} f(x)dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x)dx .$$

Since f is integrable over each subinterval $[x_{k-1}, x_k]$ there exists a partition σ_k of the subinterval, and upper and lower sums U_k , L_k such that

(1) $$0 \leq U_k - L_k < \epsilon .$$

It follows for $I_k = \int_{x_{k-1}}^{x_k} f(x)dx$ that

(2) $$0 \leq U_k - I_k < \epsilon .$$

For the partition τ of $[a,b]$ which consists of all the points of the partitions σ_k taken together, we have

$$U - L = \sum_{k=1}^{n} U_k - \sum_{k=1}^{n} L_k$$

$$= \sum_{k=1}^{n} (U_k - L_k) < n\epsilon$$

It follows, since n is fixed, that the difference between upper and lower sums for f over the whole interval $[a,b]$ can be made less than any prescribed tolerance. We conclude that f has an integral I over $[a,b]$. Furthermore, from

$$U = \sum_{k=1}^{n} U_k$$

and

$$0 \leq U - I < \epsilon$$

we have

$$0 \leq \sum_{k=1}^{n} U_k - I < \epsilon$$

and from (2)

$$0 \leq \sum_{k=1}^{n} U_k - \sum_{k=1}^{n} I_k < n\epsilon .$$

We conclude that

$$-\epsilon < I - \sum_{k=1}^{n} I_k < n\epsilon$$

or

$$|I - \sum_{k=1}^{n} I_k| < n\epsilon$$

from which the conclusion is evident. This result complements that of Theorem 6-4a. From Theorem 6-4b it follows that if f is integrable over $[a,b]$, then the integral of f can be expressed as the sum of the integrals over the intervals of a subdivision. Here we state that if the integrals over the intervals of a subdivision exist, then the integral of f over $[a,b]$ exists and is the sum of the integrals over the subintervals.

9. (a) Prove for $0 < x_{k-1} < x_k$ that

$$x_{k-1}^{2}(x_k - x_{k-1}) < \frac{x_k^{3} - x_{k-1}^{3}}{3} < x_k^{2}(x_k - x_{k-1}) .$$

Consider $u^3 - v^3$ for $0 < v < u$, we have

$$u^3 - v^3 = (u - v)(u^2 + uv + v^2) ,$$

whence

$$3(u - v)v^2 < u^3 - v^3 < 3(u - v)u^2 .$$

Setting $u = x_k$, $v = x_{k-1}$, we have the desired result.

(b) From Part (a) prove that for $0 < a < b$

$$\int_a^b x^2 \, dx = \frac{b^3}{3} - \frac{a^3}{3} .$$

Since

$$\sum_{k=1}^{n} \frac{x_k^{3} - x_{k-1}^{3}}{3} = \frac{x_n^{3} - x_0^{3}}{3} = \frac{b^3 - a^3}{3} ,$$

the result follows from the squeeze between upper and lower sums given by Part (a).

(c) Generalize this method to obtain

$$\int_a^b x^n \, dx$$

for $0 < a < b$.

Since, for $0 < v < u$

$$u^{n+1} - v^{n+1} = (u - v)(u^n + vu^{n-1} + v^2u^{n-2} + \dots + v^n)$$

we have

$$(n + 1)(u - v)v^n < u^{n+1} - v^{n+1} < (n + 1)(u - v)u^n \ .$$

It follows for the upper and lower Riemann sums for the partition $\sigma = \{x_0, x_1, \dots, x_p\}$,

$$U(\sigma) = \sum_{k=1}^{p} x_k{}^n(x_k - x_{k-1}) \ ,$$

$$L(\sigma) = \sum_{k=1}^{p} x_{k-1}^n(x_k - x_{k-1}) \ ,$$

that

$$L(\sigma) < \sum_{k=1}^{p} \frac{x_k^{n+1} - x_{k-1}^{n+1}}{n + 1} < U(\sigma) \ .$$

However, $$\frac{1}{n + 1} \sum_{k=1}^{p} (x_k^{n+1} - x_{k-1}^{n+1})$$

$$= \frac{1}{n + 1} (x_p^{n+1} - x_0^{n+1})$$

$$= \frac{1}{n + 1} (b^{n+1} - a^{n+1}) \ .$$

Solutions Exercises 6-4

1. Prove the corollary to Theorem 6-4b.

We have the result for $a < b < c$. It is necessary to consider all orders of the three points. It is clear that if the result is valid for the points in any given order, then it is valid for the reverse order, since a reversal of order reverses the signs of all the integrals. The number of cases we must consider is halved by this argument. We enumerate all remaining cases.

(a) $a \le c \le b$
(b) $b \le a \le c$
(c) $c \le b \le a$

We exhibit the proof of a .

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \left(\int_a^c f(x)dx + \int_c^b f(x)dx\right) - \int_c^b f(x)dx$$
$$= \int_a^c f(x)dx$$

2. (a) Show that $f : x \longrightarrow x^2$ is integrable over $[a,b]$, no matter what the values of a and b.

If 0 is not an interior point of $[a,b]$ then f is monotone on $[a,b]$, therefore integrable. If $a < 0 < b$ then f is monotone on $[a,0]$ and on $[0,b]$, hence integrable on both intervals. Use the result or method of Exercise 6-3, No. 8.

(b) Prove that $\int_0^a x^2\,dx = \frac{a^3}{3}$ for $a \le 0$.

Compare the approximating sums with those for $\int_0^{-a} x^2\,dx$ or sum directly. Observe that the result of b together with Example 6-4a imply for all a that

$$\int_0^a x^2\,dx = \frac{a^3}{3}.$$

(c) Prove that $\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}$ for all values of a and b.

$$\int_a^b x^2\,dx = \int_0^b x^2\,dx - \int_0^a x^2\,dx .$$

3. Exhibit the details of the proof of Lemma 6-4a when $\alpha < 0$.

If $m_k \leq f(x) \leq M_k$ on $[x_{k-1}, x_k]$ and $\alpha < 0$, then multiply by α to obtain

$$\alpha M_k \leq \alpha f(x) \leq \alpha m_k .$$

From $\alpha \sum_{k=1}^{n} M_k(x_k - x_{k-1}) \leq \alpha \sum_{k=1}^{n} m_k(x_k - x_{k-1})$ obtain the lower sum αU and upper sum αL for f over $[a,b]$. If $U - L < \epsilon$, then

$$0 < \alpha L - \alpha U < |\alpha|\epsilon .$$

Observe that αf is then integrable by Theorem 6-3a. If I is the integral of f and J that of αf over $[a,b]$, then

$$0 \leq U - I < \epsilon , \quad 0 \leq J - \alpha U < |\alpha|\epsilon ,$$

from which it follows that

$$\begin{aligned} |J - \alpha I| &= |(J - \alpha U) + \alpha(U - I)| \\ &\leq |J - \alpha U| + |\alpha| \cdot |U - I| \\ &< 2|\alpha|\epsilon . \end{aligned}$$

This result holds for all positive ϵ, hence $J = \alpha I$.

4. (a) If the graph of f is symmetric with respect to the origin, then f is odd. Prove that if f is odd and integrable on $[-a,a]$, then

$$\int_{-a}^{a} f(x)dx = 0 .$$

(b) If the graph of f is symmetric with respect to the y-axis, then f is even. Prove for an even function f which is integrable on $[-a,a]$ that

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx .$$

Interpret this result geometrically.

Take a subdivision of the interval into $2n$ equal parts and set $h = \frac{a}{n}$. Take the Riemann sum

$$R = \sum_{k=1}^{n} f(-\xi_k)h + \sum_{k=1}^{n} f(\xi_k)h .$$

where $(k - 1)h \leq \xi_k \leq kh$, $(k = 1 , \ldots , n)$.

(a) If f is odd, $f(-x) = -f(x)$, and

$$R = \sum_{k=1}^{n} [-f(\xi_k)h] + \sum_{k=1}^{n} f(\xi_k)h$$

$$= 0 .$$

Since the limit of the Riemann sums is the same independently of the method of subdivision and the choice of ξ_k it follows that the integral is 0.

(b) If f is even, $f(-x) = f(x)$, and $R = 2 \sum_{k=1}^{n} f(\xi_k)h$ where the sum is the Riemann sum for f over the half interval $[0,a]$, and the result follows on taking the limit. Geometrical interpretation: The area of a standard region is equal to that of its mirror image. Parts (a) and (b) can also be done by comparing upper and lower sums on the half-intervals.

5. Prove Theorem 6-4c as a consequence of Lemmas 6-4a and 6-4b. Conversely, derive the Lemmas as corollaries of Theorem 6-4c.

Proof of Theorem 6-4c.

Let f and g be integrable over $[a,b]$. From Lemmas 6-4a and 6-4b applied in succession

$$\alpha\int_a^b f(x)dx + \beta\int_a^b g(x)dx = \int_a^b \alpha f(x)dx + \int_a^b \beta\, g(x)dx$$

$$= \int_a^b [\alpha\, f(x) + \beta\, g(x)]dx .$$

Proof of Lemma 6-4a. Take $\beta = 0$ in Theorem 6-4c.

Proof of Lemma 6-4b. Take $\alpha = \beta = 1$ in Theorem 6-4c.

6. **Prove:** If f and g are integrable where $g : x \longrightarrow |f(x)|$ on $[a,b]$, then

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx .$$

The absolute value function is continuous; hence, for the Riemann sums R for f over $[a,b]$,

$$\left|\lim_{\nu(\sigma)\sim 0} R\right| = \lim_{\nu(\sigma)\sim 0} |R| .$$

Further,

$$|R| = \left|\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1})\right|$$

$$\le \sum_{k=1}^{n} |f(\xi_k)|(x_k - x_{k-1}) .$$

We then have for the Riemann sum

$$S = \sum_{k=1}^{n} g(\xi_k)(x_k - x_{k-1}) , \text{ where } g(\xi_k) = |f(\xi_k)| ,$$

that $|R| \le S$.

By the same argument as that of Theorem 3-4f (page 86) we have

$$\lim_{\nu(\sigma)\sim 0} |R| \le \lim_{\nu(\sigma)\sim 0} S ,$$

which is the desired result.

Alternative proof.

For all x in $[a,b]$

$$-|f(x)| \le f(x) \le |f(x)| ,$$

whence by Theorem 6-4a,

$$\int_a^b -|f(x)|dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|dx$$

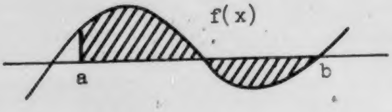
or

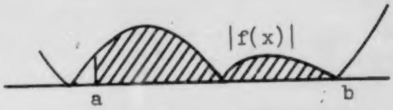
$$-\int_a^b |f(x)|dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|dx$$

or

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx \ .$$

We observe that $\int_a^b |f(x)|dx$ represents the sum of the areas of the regions bounded by the graph of f, the x-axis, and the vertical lines, $x = a$, $x = b$.





7. Compute the values of the given integrals using Theorem 6-4c.

(a) $\int_2^3 (3x^2 - 5x + 1)dx$

$$\int_2^3 (3x^2 - 5x + 1)dx = 3\int_2^3 x^2\,dx - 5\int_2^3 x\,dx + \int_2^3 dx \ .$$

From Examples 6-4a, b, c, and

$$\int_a^b f(x)dx = \int_0^b f(x)dx - \int_0^a f(x)dx$$

it follows that the integral is

$$3\left(\frac{3^3}{3} - \frac{2^3}{3}\right) - 5\left(\frac{3^2}{2} - \frac{2^2}{2}\right) + (3 - 2) = 7\frac{1}{2} \ .$$

(b) $\int_0^2 (x - 1)(x + 2)dx$

$$\begin{aligned}\int_0^2 (x - 1)(x + 2)dx &= \int_0^2 (x^2 + x - 2)dx \\ &= \frac{2^3}{3} + \frac{2^2}{2} - 2(2) \\ &= \frac{2}{3} \ .\end{aligned}$$

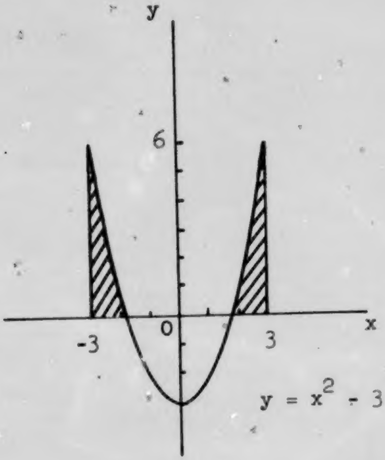
(c) $\int_{-2}^{3} (x + 2)(x - 3)dx$

$$\int_{-2}^{3} (x + 2)(x - 3)dx = \int_{-2}^{3} (x^2 - x - 6)dx$$

$$= \left(\frac{3^3}{3} - \frac{(-2)^3}{3}\right) - \left(\frac{3^2}{2} - \frac{(-2)^2}{2}\right) - 6\bigl(3 - (-2)\bigr)$$

$$= -\frac{125}{6} .$$

8. (a) Find the area of the region below the parabola $y = x^2 - 3$ above the x-axis and between the lines $x = -3$, $x = 3$.

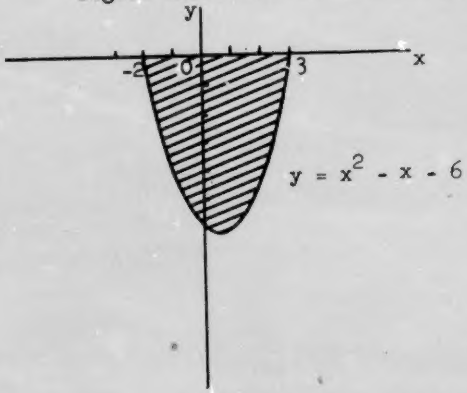


$$\text{Area} = 2\int_{\sqrt{3}}^{3} (x^2 - 3)dx$$

$$= 2\left[\left(\frac{3^3}{3} - \frac{(\sqrt{3})^3}{3}\right) - 3(3 - \sqrt{3})\right]$$

$$= 4\sqrt{3} .$$

(b) Find the area of the region between the graph of $f : x \longrightarrow x^2 - x - 6$, the x-axis, and the lines $x = -2$, $x = 3$. First draw a rough sketch of f and indicate (by shading) the region whose area is to be computed.



$$\text{Area} = -\int_{-2}^{3} (x^2 - x - 6)dx$$

$$= \frac{125}{6} .$$

(See No. 7c.)

9. Find all values of a for which

$$\int_0^a (x + x^2)dx = 0 .$$

The number a must satisfy $\frac{a^3}{3} + \frac{a^2}{2} = 0$. This equation has two solutions

$$a = -\frac{3}{2} \text{ and } a = 0 .$$

10. Compute $\int_0^3 f(x)dx$ where

$$f(x) = \begin{cases} 2 - x^2 , & 0 \le x \le 1 \\ 5 - 4x , & 1 \le x \le 3 \end{cases}$$

$$\int_0^3 f(x)dx = \int_0^1 (2 - x^2)dx + \int_1^3 (5 - 4x)dx$$

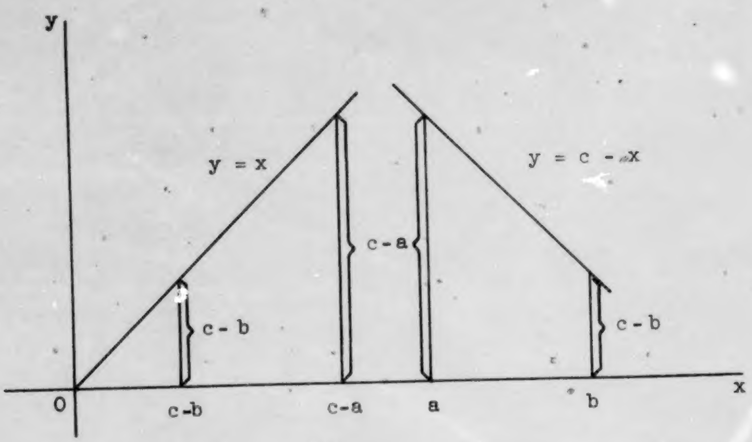
$$= (2 - \frac{1}{3}) + [10 - 4(\frac{3^2}{2} - \frac{1}{2})]$$

$$= \frac{-13}{3} .$$

11. Verify that the following property holds for $f : x \longrightarrow x$

$$\int_a^b f(c - x)dx = \int_{c-b}^{c-a} f(x)dx .$$

Explain the property geometrically in terms of areas. Do you think that the property holds for other functions that are integrable? Justify your answer.



The integrals represent areas of mirror-image regions. The property is general for a function f integrable over $[c - b, c - a]$. For $f(x) = x$, particularly,

$$\int_a^b (c - x)dx = c(b - a) - \frac{1}{2}(b^2 - a^2) ,$$

$$\int_{c-b}^{c-a} x\, dx = \frac{1}{2}[(c - a)^2 - (c - b)^2]$$

$$= \frac{1}{2}[2bc - 2ac + a^2 - b^2]$$

$$= c(b - a) - \frac{1}{2}(b^2 - a^2) .$$

12. If a function f is periodic with period λ and integrable for all x, show that

$$\int_a^{a+n\lambda} f(x)dx = n\int_a^{a+\lambda} f(x)dx$$

(n, integer). Interpret geometrically.

$$\int_a^{a+n\lambda} f(x)dx = \sum_{k=1}^{n} \int_{a+(k-1)\lambda}^{a+k\lambda} f(x)dx .$$

Now consider the subdivision of the interval $[a + (k - 1)\lambda , a + k\lambda]$ into m equal parts by means of the partition $\{u_0, u_1, \ldots, u_m\}$ where $u_i = a + (k - 1)\lambda + \frac{i\lambda}{m}$ and form the Riemann sum

$$R_k = \sum_{i=1}^{m} f(u_i)\frac{\lambda}{m}$$

$$= \sum_{i=1}^{n} f\left(a + (k - 1)\lambda + \frac{i\lambda}{m}\right)\frac{\lambda}{m}$$

$$= \sum_{i=1}^{n} f(a + \frac{i\lambda}{m})\frac{\lambda}{m} .$$

Since the Riemann sums R_k , $(k = 1 , \dots , n)$ for each of the integrals are the same, it follows that the integrals over the intervals $[a + (k - 1)\lambda , a + k\lambda]$ are equal and the result follows.

Geometrically, the standard regions for the intervals $[a + (k - 1)\lambda , a + k\lambda]$ are congruent.

13. Evaluate

$$\int_0^{100\pi} (1 + \sin 2x)dx$$

(assuming $\sin 2x$ is integrable).
Note: This exercise uses Exercises 6-2, No. 1(d) which requires Section A3-2(ii).

Since the integrand is periodic with period π , it follows from Number 12 that

$$\int_0^{100\pi} (1 + \sin 2x)dx = 100\int_0^{\pi} (1 + \sin 2x)dx .$$

From Exercises 6-2, Number 1(d)

$$\int_0^{\pi} \sin 2x \, dx = \frac{1 - \cos 2\pi}{2} = 0 .$$

Answer: 100π .

14. Prove that if f is integrable on $[a,b]$ and if $f(x) \geq 0$ for all x in $[a,b]$, then

$$\int_a^b f(x)dx \geq 0 .$$

For every partition of $[a,b]$, $L = 0$ is a lower sum. Since the integral is an upper bound for all lower sums, the result follows. The result is also an immediate consequence of Theorem 6-4a.

15. Interpret $\int_a^b f(x)dx$ in terms of area if $f(x)$ may take on both positive and negative values in $[a,b]$.

Suppose that it is possible to subdivide $[a,b]$ in such a way that in the interior of each subinterval $f(x)$ does not change sign. The integral is the sum of the areas of the standard regions under the graph of f on those subintervals in which $f(x)$ is nonnegative less the sum of the areas of the standard regions under the graph of $-f$ on those subintervals where $f(x)$ is nonpositive.

16. Interpret $\int_a^b \{g(x) - f(x)\}dx$ in terms of area if we admit the possibility that $f(x) > g(x)$ for some values of x in $[a,b]$.

The integral represents the sum of the signed areas of regions bounded above and below by the two graphs $y = g(x)$ and $y = f(x)$ and on the left and right by vertical lines $x = a$, $x = b$. If we assume there is a subdivision for which $g(x) - f(x)$ has the same sign over each subinterval, then the integral is the sum of the areas of those regions where $g(x) \geq f(x)$ less the sum of the areas of those regions where $g(x) \leq f(x)$.

17. Prove that if f and g are integrable over $[a,b]$, then

$$\left|\int_a^b \{g(x) - f(x)\}dx\right| \leq \int_a^b |g(x)|dx + \int_a^b |f(x)|dx .$$

From Number 6

$$\left|\int_a^b [g(x) - f(x)]dx\right| \leq \int_a^b |g(x) - f(x)|dx .$$

But $|g(x) - f(x)| \leq |g(x)| + |f(x)|$. It follows from Theorem 6-4a and 6-4b that

$$\int_a^b |g(x) - f(x)|dx \leq \int_a^b |g(x)|dx + \int_a^b |f(x)|dx .$$

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18. Let f and g be integrable and suppose that $f(x) \leq g(x)$ on $[a,b]$.
(a) If the inequality $f(x) + \epsilon \leq g(x)$, for some $\epsilon > 0$, holds on any subinterval of $[a,b]$, prove the strong inequality

$$\int_a^b f(x)dx < \int_a^b g(x)dx .$$

Let $[u,v]$ be a subinterval of $[a,b]$ in which $f(x) + \epsilon < g(x)$. We have $g(x) - f(x) > \epsilon$ and by Theorem 6-4a

$$\int_u^v [g(x) - f(x)]dx \geq \int_u^v \epsilon \, dx \geq \epsilon \, (v - u) .$$

Since $g(x) - f(x) \geq 0$ on the rest of the interval we have

$$\int_a^b [g(x) - f(x)]dx = \int_a^u [g(x) - f(x)]dx + \int_u^v [g(x) - f(x)]dx + \int_v^b [g(x) - f(x)]dx$$

$$\geq 0 + \epsilon \, (v - u) + 0$$

whence

$$\int_a^b g(x)dx - \int_a^b f(x)dx \geq \epsilon \, (v - u) > 0$$

from which the conclusion follows.

(b) If f and g are continuous at $x = u$ in $[a,b]$ and $f(u) < g(u)$ prove that strong inequality holds as above.

If f and g are continuous at u then $g - u$ is continuous at u and, under the conditions of the problem

$$g(u) - f(u) > 0 .$$

From Lemma 3-4 it follows that there is a neighborhood of u in which

$$g(x) - f(x) > \epsilon > 0$$

where $0 < \epsilon < g(u) - f(u)$. The result follows from Part (a).

19. If functions f and g are integrable, and $f(x) \leq h(x) \leq g(x)$ on $[a,b]$, does it follow that

$$\int_a^b f(x)dx \leq \int_a^b h(x)dx \leq \int_a^b g(x)dx \ ?$$

Illustrate by an example.

No! The function h may not be integrable. For example, take $f(x) = -1$, $g(x) = 1$,

$$h(x) = \begin{cases} 1, & x \text{ rational} \\ -1, & x \text{ irrational} . \end{cases}$$

On every interval $\max\{h(x)\} = 1$, $\min\{h(x)\} = -1$, hence for every upper sum, $U \geq (b - a)$, and for every lower sum, $L \leq -(b - a)$. Thus h is not integrable by Theorem 6-3a.

20. (a) Prove the Mean Value Theorem of integral calculus: If f is continuous and integrable on $[a,b]$, then there exists a value u in the open interval (a,b) such that

$$\int_a^b f(x)dx = f(u)(b - a) .$$

By the Extreme Value Theorem (Theorem 3-7b) $f(x)$ takes on a maximum value M and a minimum value m in $[a,b]$. Since

$$m \leq f(x) \leq M$$

on $[a,b]$ we have from Theorem 6-4a

$$\int_a^b m\,dx \leq \int_a^b f(x)dx \leq \int_a^b M\,dx$$

whence

$$m \leq \frac{1}{b - a} \int_a^b f(x)dx \leq M .$$

Since $f(x)$ takes on every value between m and M (Intermediate Value Theorem, Theorem 3-7a), there is a value u in $[a,b]$ for which

$$f(u) = \frac{1}{b - a} \int_a^b f(x)dx .$$

(b) Show that the value $f(u)$ in (a) satisfies

$$f(u) = \lim_{h\sim 0} \frac{f_0 + f_1 + \dots + f_n}{n + 1}$$

where $h = \frac{(b - a)}{n}$ and $f_k = f(a + kh)$ for $k = 0, 1, 2, \dots, n$. Thus $f(u)$ can be interpreted as an extension of the idea of mean or arithmetic average to the values of a function on an interval.

The expression for the integral as a limit of Riemann sums is

$$\int_a^b f(x)dx = \lim_{h\sim 0} \sum_{k=1}^{n} f_k\, h = \lim_{h\sim 0} \sum_{k=0}^{n} f_k\, h$$

$$= \lim_{h\sim 0} (n + 1)h \left\{\frac{\sum_{k=0}^{n} f_k}{n + 1}\right\}$$

$$= \lim_{h\sim 0} (b - a + h) \left\{\frac{\sum_{k=0}^{n} f_k}{n + 1}\right\}$$

$$= \lim_{h\sim 0} (b - a + h) \lim_{h\sim 0} \left\{\frac{\sum_{k=0}^{n} f_k}{n + 1}\right\}$$

from which the result follows (provided

$$\lim_{h\sim 0} \left\{\frac{\sum_{k=0}^{n} f_k}{n + 1}\right\}$$

exists.) Existence is a consequence of the fact that if $\lim_{h\sim 0} pq$ exists and $\lim_{h\sim 0} p$ exists but $\lim_{h\sim 0} p \neq 0$ then $\lim_{h\sim 0} q$ exists and

$$\lim_{h\sim 0} q = \frac{\lim_{h\sim 0} pq}{\lim_{h\sim 0} p}.$$

Although this is not the same kind of limit as that of Corollary 1 to Theorem 3-4e the result and proof are essentially the same. (The point need not be brought up unless the issue of existence is raised in class.)

21. If $\frac{a_0}{n+1} + \frac{a_1}{n} + \dots + \frac{a_{n-1}}{2} + \frac{a_n}{1} = 0$, show that
$$a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$$
has at least one real root in $(0,1)$.

Set $f(x) = \sum_{k=0}^{n} a_kx^{n-k}$. By the Mean Value Theorem (No. 20a) there is a point u in $(0,1)$ for which, by Exercises 6-3, Number 9(c),

$$\begin{aligned} f(u) &= \int_0^1 f(x)dx \\ &= \frac{a_0x^{n+1}}{n+1} + \frac{a_1x^n}{n} + \dots + \frac{a_nx}{1} \Bigg|_0^1 \\ &= \frac{a_0}{n+1} + \frac{a_1}{n} + \dots + \frac{a_n}{1} \\ &= 0 . \end{aligned}$$

22. (a) Prove that if $f(x)$ is integrable on $[a,b]$, then $|f(x)|$ is integrable on $[a,b]$.

For each positive ϵ there exist a partition $\sigma = \{x_1, x_2, x_3, \dots, x_n\}$, an upper sum U , and a lower sum L over σ for which

$$U - L = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) < \epsilon ,$$

where $M_k \le f(x) \le m_k$ on $[x_{k-1}, x_k]$. We shall describe upper and lower bounds M_k^* and m_k^* for $|f(x)|$ on each interval $[x_{k-1}, x_k]$. If $m_k \geq 0$, then $|f(x)| = f(x)$ and we take $M_k^* = M_k$ and $m_k^* = m_k$ on $[x_{k-1}, x_k]$. If $M_k \le 0$, then $|f(x)| = -f(x)$ and we take $M_k^* = -m_k$ and $m_k^* = -M_k$; whence

$$M_k^* - m_k^* = -m_k - (-M_k) = M_k - m_k .$$

If $M_k \geq 0$ and $m_k < 0$ we consider two cases:

(i) $|m_k| \le M_k$. Taking $M_k^* = M_k$ and $m_k^* = 0$, we have

$$M_k^* - m_k^* = M_k < M_k + |m_k| \le M_k - m_k \ .$$

(ii) $|m_k| > M_k$. Taking $M_k^* = |m_k|$ and $m_k^* = 0$, we have

$$M_k^* - m_k^* = |m_k| \le |m_k| + M_k \le M_k - m_k \ .$$

In every case

$$U^* - L^* \le U - L < \epsilon$$

where U^* and L^* are the upper and lower sums for $|f(x)|$ constructed by use of the bounds M_k^* and m_k^* .

Alternate Solution.

Define f^+ and f^- by

$$f^+(x) = \begin{cases} f(x) \ , & \text{if } f(x) \ge 0 \ , \\ 0 \ , & \text{if } f(x) \le 0 \ , \end{cases} \qquad f^-(x) = \begin{cases} -f(x) \ , & \text{if } f(x) \le 0 \ , \\ 0 \ , & \text{if } f(x) \ge 0 \ . \end{cases}$$

Since $|f| = f^+ + f^-$, it is necessary only to show that f^+ and f^- are integrable in order to prove that $|f|$ is integrable. Now, for each positive ϵ , there exist a partition σ , an upper sum U , and a lower sum L for f over σ such that, in the notation of the text,

$$U - L = \sum (M_k - m_k)(x_k - x_{k-1}) < \epsilon \ .$$

On each subinterval $[x_{k-1}, x_k]$ choose upper and lower bounds M^+ and m^+ for f^+ as follows. If $f(x) \ge 0$ for at least one point x in $[x_{k-1}, x_k]$ take $M_k^+ = M_k$ and $m_k^+ = m_k$; if $f(x) < 0$ everywhere on the interval, take $M_k^+ = m_k^+ = 0$. In either case

$$M_k^+ - m_k^+ \le M_k - m_k \ .$$

Similarly, for f^- the corresponding upper and lower bounds satisfy

$$M_k^- - m_k^- \le M_k - m_k \ .$$

It follows for the corresponding upper and lower sums that

$$U^+ - L^+ < \epsilon \ , \ U^- - L^- < \epsilon \ .$$

Consequently, f^+ and f^- are integrable and so also is $|f|$.

(b) Show that the converse is not true.

Consider the function defined by

$$f(x) = \begin{cases} 1 \ , & x \text{ rational} \\ -1 \ , & x \text{ irrational} . \end{cases}$$

We have $|f(x)| = 1$ for all x , therefore $|f(x)|$ is integrable. For f itself, the maximum value 1 and the minimum value -1 are taken on in every interval. Consequently, $M_k \geq 1$, $m_k \leq -1$ and $M_k - m_k \geq 2$. It follows that for every upper sum U and lower sum L that $U - L \geq 2(b - a)$.

23. If f and g are integrable on $[a,b]$ then both $\max(f,g)$ and $\min(f,g)$ are also integrable on $[a,b]$.

Use Number 22 and the results of Exercises A1-3, Number 7:

$$\max(f,g) = \frac{1}{2}(f + g + |f - g|) ,$$

$$\min(f,g) = \frac{1}{2}(f + g - |f - g|) ;$$

the result follows from Lemma 6-4b.

Alternate Solution.

There exists a partition σ and upper and lower sums U_1 and L_1 for f , U_2 and L_2 for g , over σ such that both

$$U_1 - L_1 < \epsilon \quad \text{and} \quad U_2 - L_2 < \epsilon .$$

(For example, if σ_1 is such a partition for f , σ_2 for g , take $\sigma = \sigma_1 \cup \sigma_2$.) Let M_k' , m_k' denote the bounds for f in the expressions for U_1 and L_1 , M_k'' and m_k'' the corresponding bounds for g . Set $\Phi = \max(f,g)$. On each interval $[x_{k-1},x_k]$ choose as the upper bound for Φ , $M_k = \max(M_k',M_k'')$; as the lower bound, choose

$$m_k = \begin{cases} m_k' \ , & \text{if } M_k = M_k' , \\ m_k'' \ , & \text{otherwise} . \end{cases}$$

With this choice,

$$\begin{aligned} M_k - m_k &\leq \max(M_k' - m_k' \ , \ M_k'' - m_k'') \\ &\leq (M_k' - m_k') + (M_k'' - m_k'') . \end{aligned}$$

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Form the appropriate upper and lower sums U and L for Φ to obtain

$$U - L \leq (U_1 - L_1) + (U_2 - L_2) < 2\epsilon$$

from which the conclusion follows.

To obtain the result for $\min(f,g)$ observe that $\min(f,g) = -\max(-f,-g)$.

Finally, since $|f| = \max(f,-f)$, this proof also serves to demonstrate Number 22.

24. Let f and g be bounded and integrable on $[a,b]$.
(a) Prove $f \cdot g$ is integrable on $[a,b]$.

First observe that it is sufficient to prove the result when f and g are positive. This is true from the boundedness of f and g , because there exist constants c_1 and c_2 such that $\phi = f + c_1$ and $\psi = g + c_2$ are positive. If the result is true for positive functions then $\phi \cdot \psi = f \cdot g + c_1 g + c_2 f + c_1 c_2$ is integrable, hence $f \cdot g$ is integrable.

Now let σ be a partition, M_k' and m_k' upper and lower bounds for f , M_k'' and m_k'' for g , on the subinterval $[x_{k-1},x_k]$ of the partition, and let U' , L' and U'' , L'' be the corresponding upper and lower sums. For a sufficiently fine partition and appropriate choice of bounds,

$$U' - L' < \epsilon \text{ and } U'' - L'' < \epsilon .$$

Now, choose the upper and lower bounds $M_k = M_k' M_k''$, $m_k = m_k' m_k''$ for $f \cdot g$. Then

$$\begin{aligned} M_k - m_k &= M_k'(M_k'' - m_k'') + m_k''(M_k' - m_k') \\ &\leq A(M_k'' - m_k'') + B(M_k' - m_k') \end{aligned}$$

where A and B are overall upper bounds for f and g , respectively. Form the appropriate upper and lower sums for $f \cdot g$ over σ to obtain

$$\begin{aligned} U - L &\leq A(U' - L') + B(U'' - L'') \\ &< \epsilon(A + B) , \end{aligned}$$

from which it follows that $f \cdot g$ is integrable.

(b) If g is bounded away from zero, then $\frac{f}{g}$ is integrable over $[a,b]$.

For the proof, it is sufficient to prove $\frac{1}{g}$ is integrable and then to apply Part (a). Now suppose $|g(x)| \geq c > 0$. Let U and L be upper and lower sums for g over σ such that $U - L < \epsilon$, and let M_k, m_k denote upper and lower bounds for g on the subinterval $I_k = [x_{k-1}, x_k]$. Choose upper and lower bounds M_k^* and m_k^* for $\frac{1}{g}$ as follows. If $m_k \geq c$ or $M_k \leq -c$, then take $M_k^* = \frac{1}{m_k}$ and $m_k^* = \frac{1}{M_k}$. In this case, $M_k^* - m_k^* = \frac{M_k - m_k}{m_k M_k} \leq \frac{M_k - m_k}{c^2}$. If $m_k \leq -c$ and $M_k \geq c$, then take $m_k^* = -\frac{1}{c}$ and $M_k^* = \frac{1}{c}$. In this case use $1 \leq \frac{M_k}{c}$ and $1 \leq -\frac{m_k}{c}$ to obtain $M_k^* - m_k^* = \frac{1}{c} + \frac{1}{c} \leq \frac{M_k}{c^2} - \frac{m_k}{c^2} \leq \frac{M_k - m_k}{c^2}$. For the corresponding upper and lower sums U^* and L^*, then

$$U^* - L^* \leq \frac{U - L}{c^2} < \frac{\epsilon}{c^2}.$$

25. If f and g are bounded and integrable, then $\int_a^b (\alpha f(x) + \beta g(x))^2 dx$ exists and is greater than or equal to 0 for all constant α and β. Show from this that

$$\int_a^b f(x)^2 dx \int_a^b g(x)^2 dx \geq \left\{ \int_a^b f(x)g(x)dx \right\}^2$$

with equality if and only if (for f and g continuous) $f = cg$ on $[a,b]$ for some constant c.

Consider the inequality

$$\int_a^b (f(x) + tg(x))^2 dx = \int_a^b f(x)^2 dx + 2t\int_a^b f(x)g(x)dx + t^2 \int_a^b g(x)^2 dx$$
$$\geq 0.$$

From the solution of Exercises A1-2, Number 15, an inequality of the form $At^2 + 2Bt + C > 0$ holds for all t if and only if $B^2 - AC \leq 0$, but that is precisely Buniakowsky's inequality in this case. (Compare the solution of Exercises A1-2, No. 16.)

If $f = cg$ then equality obviously holds. If equality holds $B^2 - AC = 0$, then for some choice of t, say $t = -c$

$$\int_a^b [f(x) - cg(x)]^2 dx = 0 .$$

Now $f - cg$ must be identically zero, for if there were any point where $f(x_0) - cg(x_0) \neq 0$, then $[f(x_0) - cg(x_0)]^2 > 0$ and from the continuity of f and g there would be an interval containing x_0 where the integrand has a positive lower bound. In that case the integral would have to be positive, not zero as required. Consequently, $f = cg$ is the only possibility. (Note that the proof requires the continuity of f and g at only one point.)

26. If f is integrable and its graph is flexed upward on the interval $[0,a]$, show that

$$\int_0^a f(x)dx \geq a\, f(\tfrac{a}{2}) .$$

Interpret geometrically.

The graph of f lies above its tangent at $\frac{a}{2}$:

$$f(x) \geq f(\tfrac{a}{2}) + f'(\tfrac{a}{2})(x - \tfrac{a}{2})$$

(if f is not differentiable there still exists a "line of support" at $\frac{a}{2}$ and $f'(\frac{a}{2})$ would be replaced by the slope m of a line of support). Then

$$\int_0^a f(x)dx \geq \int_0^a [f(\tfrac{a}{2}) + f'(\tfrac{a}{2})(x - \tfrac{a}{2})]dx$$

$$\geq \left[f(\tfrac{a}{2})x + f'(\tfrac{a}{2})\left(\frac{x^2}{2} - \frac{ax}{2}\right)\right] \Bigg|_0^a$$

$$\geq a\, f(\tfrac{a}{2}) .$$

For f positive, the geometrical intepretation is that the area of the standard region under the graph of f is greater than the area under the tangent taken at the midpoint of the base interval.

27. Show that

$$\sqrt{(a^2 + \tfrac{1}{3})(b^2 + \tfrac{1}{3})} \geq \int_0^1 \sqrt{(x^2 + a^2)(x^2 + b^2)}\, dx$$

Set $f(x) = \sqrt{x^2 + a^2}$, $g(x) = \sqrt{x^2 + b^2}$ in the Buniakowsky inequality (No. 25).

28. Show that

(a) $\frac{1}{2} + \frac{3\sqrt{2}}{8} < \int_0^1 \sqrt{1 + x^3}\, dx < \frac{\sqrt{5}}{2}$.

Take $f(x) = 1$, $g(x) = \sqrt{1 + x^3}$ in the Buniakowsky inequality to obtain the upper estimate. For the lower estimate, use

$$\begin{aligned} \int_0^1 \sqrt{1 + x^3}\, dx &= \int_0^{1/2} \sqrt{1 + x^3}\, dx + \int_{1/2}^1 \sqrt{1 + x^3}\, dx \\ &> \int_0^{1/2} 1\, dx + \int_{1/2}^1 \sqrt{1 + \tfrac{1}{8}}\, dx . \end{aligned}$$

(b) Show that $\frac{1}{2} + \frac{\sqrt{2}}{3} > \int_0^1 \frac{dx}{\sqrt{1 + x^3}} > \frac{2\sqrt{5}}{5}$.

For the upper estimate, use

$$\begin{aligned} \int_0^1 \frac{dx}{\sqrt{1 + x^3}} &= \int_0^{1/2} \frac{dx}{\sqrt{1 + x^3}} + \int_{1/2}^1 \frac{dx}{\sqrt{1 + x^3}} \\ &< \int_0^1 1\, dx + \int_{1/2}^1 \frac{dx}{\sqrt{1 + \frac{1}{8}}} . \end{aligned}$$

For the lower estimate use the Buniakowsky inequality:

$$\int_0^1 \sqrt{1 + x^3}\, dx \int_0^1 \frac{dx}{\sqrt{1 + x^3}} > 1 ,$$

hence, from Part (a),

$$\frac{\sqrt{5}}{2} \int_0^1 \frac{dx}{\sqrt{1 + x^3}} > 1 .$$

29. Find a continuously differentiable function F in $[0,1]$ such that
(a) $F(0) = 0$, $F(1) = a$,
(b) $\int_0^1 F(x)^2 dx = \frac{a^2}{3}$,
(c) $\int_0^1 F'(x)^2 dx$ is a minimum.

Take $f(x) = 1$ and $g(x) = F'(x)$ in the Buniakowsky inequality (No. 25):

$$\int_0^1 1\, dx \int_0^1 F'(x)^2\, dx \geq \left\{ \int_0^1 F'(x) dx \right\}^2$$

$$\geq \{F(1) - F(0)\}^2$$

$$\geq a^2 .$$

Equality holds when $g = cf$ for constant c . Thus $F'(x) = c$, hence $F(x) = cx + d$. From condition (a), $F(x) = ax$; condition (b) is automatically satisfied and is therefore redundent. Condition (c) is satisfied since equality is achieved in the inequality above.

Solutions Exercises 6-5

1. Find the volume of the solid of revolution generated by $f : x \longrightarrow \sqrt{x}$ on $[0,1]$.

The volume is $\frac{\pi}{2}$.

2. Use the procedure of this section to find the volume of a right circular cone of altitude h and base of radius r.

The cone is the solid of revolution obtained by rotating about the x-axis the standard region under the graph of $f : x \longrightarrow \frac{r}{h} x$ on $[0,h]$. The volume is

$$\pi \int_0^h \frac{r^2 x^2}{h^2} dx = \frac{\pi}{3} r^2 h .$$

3. Obtain the formula for the volume of a sphere of radius r by first showing that the sphere is a solid of revolution.

The sphere of radius r about the origin is the set of points (x,y,z) satisfying

$$x^2 + y^2 + z^2 \leq r^2 .$$

This is identical with the set

$$\{(x,y,z) : -r \leq x \leq r , y^2 + z^2 \leq (\sqrt{r^2 - x^2})^2\}$$

which is the form in which we have described a solid of revolution. The volume is given by

$$\pi \int_{-r}^{r} (r^2 - x^2)dx = \pi\{r^2[r - (-r)] - [\frac{r^3}{3} - \frac{(-r)^3}{3}]\}$$
$$= \frac{4}{3} \pi r^3 .$$

4. Find the volume of the ellipsoid generated by rotating the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its major axis. (Assume $a > b$.)

This is the solid of revolution generated by rotating about the x-axis the standard region under the graph of $y = b\sqrt{1 - \frac{x^2}{a^2}}$ on $[-a,a]$. The volume is

$$\pi \int_{-a}^{a} b^2(1 - \frac{x^2}{a^2})dx = 2\pi\int_0^a b^2(1 - \frac{x^2}{a^2})dx$$

$$= 2\pi b^2 (a - \frac{a^3}{3a^2})$$

$$= \frac{4}{3}\pi ab^2 .$$

Note that for $a = b$ the ellipsoid is a sphere and the result of Number 3 is obtained.

5. Find the volume of the segment of a sphere of radius r bounded by two parallel planes if the bases of the segment are at distances a and b from the center and are on the same side.

We assume $a < b$. As in Example 6-5a we obtain for the volume

$$\pi\int_a^b (r^2 - x^2)dx = \pi[r^2(b - a) - \frac{b^3 - a^3}{3}]$$

$$= \frac{\pi}{3}(b - a)[3r^2 - (a^2 + ab + b^2)] .$$

6. If the acceleration of a particle moving along a line is $3t - 2$ in centimeters per second per second at any time t in seconds and if the velocity is 2 centimeters per second when $t = 0$, find the distance covered during the first second.

For the velocity, we have

$$v - 2 = \int_0^t (3u - 2)du$$

so that

$$v = \frac{3}{2}t^2 - 2t + 2 .$$

For the distance d covered in the first second we have

$$\begin{aligned} d &= \int_0^1 (\tfrac{3}{2}t^2 - 2t + 2)dt \\ &= \frac{1}{2} - 1 + 2 \\ &= \frac{3}{2} \text{ in centimeters.} \end{aligned}$$

7. A particle moves along a line so that its velocity at any time is given by $v = 4t^2 - 14t + 6$. Find the distance covered by the particle between the instants when it is at rest.

The particle is at rest when $v = 0$ or $4t^2 - 14t + 6 = 2(2t-1)(t-3) = 0$; that is, when $t = \frac{1}{2}$ or $t = 3$. The displacement is

$$\begin{aligned} \int_{1/2}^{3} (4t^2 - 14t + 6)dt &= \frac{4}{3}(27 - \frac{1}{8}) - 7(9 - \frac{1}{4}) + 6(3 - \frac{1}{2}) \\ &= \frac{-125}{12} . \end{aligned}$$

The sign indicates a displacement in the negative sense along the line. The distance covered is the absolute value of the displacement; that is, $\frac{125}{12}$.

8. With what upward velocity must a ball be thrown if it is to reach a height of 100 feet above the point from which it is thrown?

In Equation (5) of the text we have $z_0 = 0$ (initial ground level) so that the height is given by

$$z = v_0 t - 16t^2 .$$

The maximum height occurs when

$$\frac{dz}{dt} = v_0 - 32t$$

is zero. Thus, we ask that

$$Z_{max} = v_0\left(\frac{v_0}{32}\right) - 16\left(\frac{v_0}{32}\right)^2 \geq 100$$

or

$$\frac{v_0^2}{32} - \frac{1}{2}\frac{v_0^2}{32} = \frac{v_0^2}{64} \geq 100$$

or

$$v_0^2 \geq 6400 .$$

Since v_0 is positive the condition is

$$v_0 \geq 80 \qquad \text{(in feet per second).}$$

9. A stone is thrown down from the top of a 200 foot tower with an initial velocity of 40 feet per second. How long will it take the stone to reach the ground? With what velocity will it hit the ground?

Under the conditions of the problem take $z_0 = 200$ and $v_0 = -40$ in Equation (5):

$$\begin{aligned} z &= 200 - 40t - 16t^2 \\ &= -8(2t - 5)(t + 5) . \end{aligned}$$

The time $t > 0$ when the stone reaches the ground $(z = 0)$ is $t = \frac{5}{2}$ (in seconds). At that time

$$v = -40 - 32t = -120 \qquad \text{(in feet per second).}$$

10. Answer the questions asked in Number 9 in the case where the stone is dropped from the tower.

If the stone is dropped we have $v_0 = 0$, or

$$z = 200 - 16t^2 .$$

The stone reaches the ground $z = 0$ when

$$t = \sqrt{\frac{200}{16}} = \frac{5\sqrt{2}}{2} \qquad \text{(in seconds)}$$

and then

$$v = -32\left(\frac{5\sqrt{2}}{2}\right) = -80\sqrt{2} \approx -113 \qquad \text{(in feet per second).}$$

11. If the stone is thrown straight up with a velocity of 40 feet per second from the top of the 200 foot tower, with what velocity will it strike the ground?

Proceed as before, to obtain

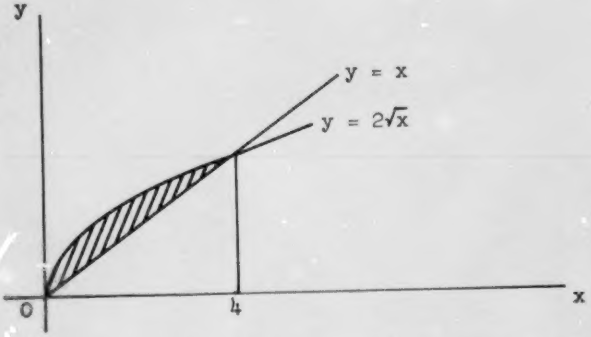
$$z = 200 + 40t - 16t^2$$
$$= -8(t - 5)(2t + 5) .$$

The stone reaches the ground $(z = 0)$ at $t = 5$ and the velocity is then

$$v = 40 - 32t = -120 \qquad \text{(in feet per second).}$$

This result is consistent with the result of Number 9, since the stone returns to its initial elevation with the negative of its initial velocity.

12. Find the volume of the solid obtained by revolving the region bounded by the parabola $y^2 = 4x$ and the line $y = x$ about the x-axis.



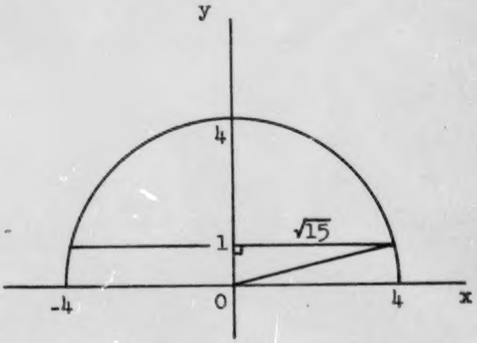
This is the difference between the volumes of the solids of revolution generated by the functions $f : x \longrightarrow 2\sqrt{x}$ and $g : x \longrightarrow x$ on $[0,4]$. The volume is

$$\pi\int_0^4 f(x)^2dx - \pi\int_0^4 g(x)^2dx = \pi\int_0^4 (4x - x^2)dx$$
$$= \pi[2(4^2) - \frac{4^3}{3}]$$
$$= \frac{32\pi}{3} .$$

13. A cylindrical hole of radius 1 inch is drilled out along a diameter of a solid sphere of radius 4 inches. Find the volume of the material cut out.

The desired volume is that of the solid of revolution generated by rotating the standard region under the graph of f on $[-4,4]$ about the x-axis, where

$$f(x) = \begin{cases} 1 \text{ , if } |x| \leq \sqrt{15} \text{ ,} \\ \sqrt{16 - x^2} \text{ , if } \sqrt{15} \leq |x| \leq 4 \text{ .} \end{cases}$$



The volume is

$$2\pi \int_0^4 f(x)^2 dx = 2\pi \int_0^{\sqrt{15}} dx + 2\pi \int_{\sqrt{15}}^4 (16 - x^2) dx$$

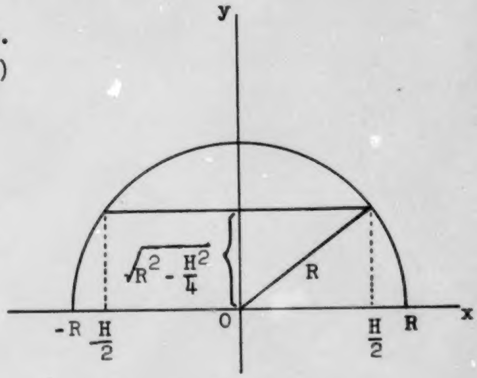
$$= 2\pi\sqrt{15} + 2\pi[\frac{128}{3} - (16\sqrt{15} - \frac{15\sqrt{15}}{3})]$$

$$= \frac{4\pi}{3}(64 - 15\sqrt{15}) \text{ .}$$

Here we have used the evenness of f .

14. Find the volume of the portion of a sphere remaining after a cylindrical hole is drilled out along its diameter if the length of the hole is H . Check your answer by considering some special cases.

Let R be the radius of the sphere. (The result will not depend on R .) The radius of the cylinder is then $\sqrt{R^2 - \frac{H^2}{4}}$. The volume is that of the solid of revolution obtained by rotating the region between the the graphs $y = \sqrt{R^2 - \frac{H^2}{4}}$ and $y = \sqrt{R^2 - x^2}$ about the x-axis, for $|x| \leq \frac{H}{2}$; namely, by the even symmetry,



$$V = 2\pi\int_0^{H/2}(R^2 - x^2)dx - 2\pi\int_0^{H/2}(R^2 - \frac{H^2}{4})dx$$

$$= \frac{\pi H^3}{6} .$$

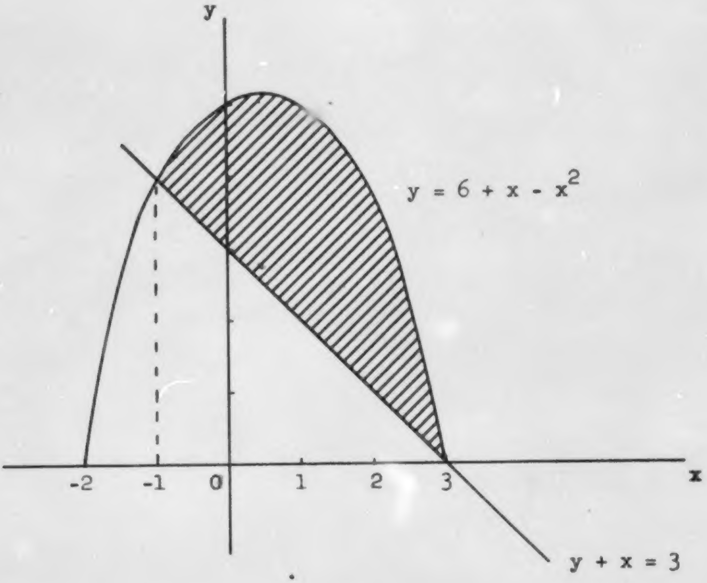
In the extreme cases, for $\frac{H}{2} = R$ (cylinder of zero radius), $V = \frac{4}{3}\pi R^3$ which is correct for the volume of the sphere; for $H = 0$ (cylinder of radius r), $V = 0$ which is correct when the sphere is completely cut away.

Solutions Miscellaneous Exercises

1. If K is a constant prove that $\int_{\sin^2 t}^{\cos^2 t} K \, dx = K \cos 2t$.

$$\int_{\sin^2 t}^{\cos^2 t} K \, dx = K(\cos^2 t - \sin^2 t) = K \cos 2t .$$

2. Find the area of the region bounded by the graph of $f : x \longrightarrow 6 + x - x^2$ and the line $y + x = 3$.



$$\int_{-1}^{3} [(6 + x - x^2) - (3 - x)]dx = \int_{-1}^{3} (3 + 2x - x^2)dx$$

$$= \frac{32}{3}$$

55/a

3. Over what intervals is the given function integrable?
   (a) $f : x \longrightarrow \frac{1}{x}$
   (b) $g : x \longrightarrow \frac{|x|}{x}$

Any closed interval excluding the origin, $x = 0$. The functions are not defined at $x = 0$ and the criterion for integrability over the interval $[a,b]$ includes the requirement that the interval be in the domain of the function.

4. For which of the functions of Number 3 is it possible to extend the domain so that the function defined on the extended domain is integrable over every closed interval? Does the value of the integral of the extended function over any interval which contains a point of the extension depend upon the way in which the domain is extended? In the light of your conclusion suggest a generalization of the definition of integral.

If we extend g by defining $g^*(0) = 0$ and $g^*(x) = g(x)$ when $x \neq 0$ (thus $g^*(x) = \operatorname{sgn} x$), then g^* is weakly increasing and therefore integrable on every closed interval. (Compare Exercises 6-3, No. 6b.) We cannot extend the definition of f to $x = 0$ so that the extended function is integrable because f is unbounded in any deleted neighborhood of the origin and therefore upper and lower sums cannot be defined.

It does not matter how we extend the function g to $x = 0$. It would still be integrable since the extended function would be a step function and the value of the integral would not be affected by the choice of the value of the extended function at $x = 0$. (Exercises 6-3, No. 6a.)

We could define the generalized integral of f on $[a,b]$, where $[a,b]$ need not be wholly in the domain of f as the integral of any extension f^* of f to the whole of $[a,b]$ provided all such extensions have the same integral.

5. The area of the standard region of a function f over $[a,b]$ is given by
$$\frac{1}{6}(2a^3 - 2b^3 - 9a^2 + 9b^2) .$$
Find a function f for which this is true. Is there more than one such function? Interpret your answer geometrically.

The expression is easily recognized as

$$\int_a^b (3x - x^2)dx$$

so that

$$f(x) = 3x - x^2$$

is one answer. However, infinitely many answers are possible. For example, the constant function

$$x \longrightarrow \frac{1}{6}[9(b + a) - 2(b^2 + ab + a^2)](b - a)$$

representing the rectangle with base $b - a$ of the same area. Furthermore, if g is any nonnegative function with

$$\int_a^b g(x)dx < \int_a^b f(x)dx$$

then by adding a suitable positive constant, $h(x) = g(x) + c$, we can translate the graph of g upward and obtain an equal area to that of the standard region of f. We need only take

$$c = \frac{1}{b - a}\int_a^b [f(x) - g(x)]dx .$$

6. A point moves on a line such that after t seconds its velocity is $v = t^2 - 9t + 20$. If its position is $s = 0$ when $t = 0$, how far does the point move during the time when the velocity is nonpositive?

We have

$$v = (t - 4)(t - 5)$$

so that v is negative for $4 < t < 5$. The initial value of s doesn't matter; we have

$$\begin{aligned} s(5) - s(4) &= \int_4^5 (t^2 - 9t + 20)dt \\ &= \frac{5^3 - 4^3}{3} - \frac{9}{2}(5^2 - 4^2) + 20(5 - 4) \\ &= -\frac{1}{6} . \end{aligned}$$

The distance is $\frac{1}{6}$ (in the negative orientation).

7. Show that

(a) $$1 < \int_0^1 \frac{x^2 + a^2}{x^2 + 1}\,dx < a^2 , \qquad (a > 1)$$

Use

$$\frac{x^2 + a^2}{x^2 + 1} > \frac{x^2 + 1}{x^2 + 1} \geq 1 ,$$

and

$$\frac{x^2 + a^2}{x^2 + 1} = 1 + \frac{a^2 - 1}{x^2 + 1} \leq 1 + \frac{a^2 - 1}{0 + 1} \leq a^2 .$$

(b) $$1 < \int_0^1 \frac{x^4 + x^2 + a^2}{x^4 + x^2 + 1}\,dx < a^2 , \qquad (a > 1) .$$

Use

$$\frac{x^4 + x^2 + a^2}{x^4 + x^2 + 1} > 1$$

and

$$\frac{x^4 + x^2 + a^2}{x^4 + x^2 + 1} = \frac{x^4 + x^2 + 1 + (a^2 - 1)}{x^4 + x^2 + 1} = 1 + \frac{a^2 - 1}{x^4 + x^2 + 1}$$

$$\leq 1 + (a^2 - 1) \leq a^2 .$$

8. For a monotone function f on $[a,b]$ we have found that the upper and lower sums over a partition σ satisfy

$$U - L < \epsilon$$

provided the norm $\nu(\sigma)$ is sufficiently small,

$$\nu(\sigma) < \frac{\epsilon}{|f(b) - f(a)|}$$

when $f(b) \neq f(a)$. Since the integral I of f over $[a,b]$ lies between L and U we can estimate I by either sum within the tolerance ϵ . Show that the arithmetic mean $\frac{1}{2}[L + U]$ is an estimate of I within the finer tolerance $\frac{\epsilon}{2}$.

If $L \leq I \leq U$, then

$$-\frac{1}{2}(U - L) \leq I - \frac{1}{2}(U + L) \leq \frac{1}{2}(U - L)$$

whence

$$-\frac{\epsilon}{2} < I - \frac{1}{2}(U + L) < \frac{\epsilon}{2} .$$

9. Find Riemann sums differing by less than 0.1 for the area of the standard region of $f : x \longrightarrow \frac{-1}{x - 1}$ over $[-1,0]$.

The function f is monotone increasing in the stated interval. It follows that the difference between the upper and lower Riemann sums for a subdivision of the interval into n equal parts satisfies

(1) $$U - L = \frac{f(0) - f(-1)}{n} = \frac{1}{2n} .$$

From the preceding exercise,

(2) $$\frac{1}{2}(U + L)$$

approximates the integral with an error no greater than $\frac{1}{4n}$. To satisfy

$$\frac{1}{4n} < \frac{1}{10}$$

take $n = 3$. For $n = 3$,

$$U = [f(-\tfrac{2}{3}) + f(-\tfrac{1}{3}) + f(0)]\tfrac{1}{3}$$
$$= \frac{3}{5} + \frac{3}{4} + 1 = 2.35 ,$$

and from (1)

$$L = U - \frac{1}{6} .$$

From (2), estimate the integral I by

$$I \approx U - \frac{1}{12} .$$

Since this differs from the integral by no more than $\frac{1}{12}$, round off the decimal expression for $\frac{1}{12}$ at the hundreths place to keep the error within the tolerance 0.1 .

Take $\frac{1}{12} \approx 0.08$ to obtain

$$I \approx 2.27 .$$

(Note that in scientific measurements, the tolerance is often stated in the following way,

$$I = 2.27 \pm 0.1 .$$

Emphasize that we do not use the notation--for us it would mean "either $I = 2.37$ or $I = 2.17$"--instead we would write

$$|I - 2.27| < 0.1 .)$$

10. If when one applies the brakes of an automobile a constant decelaration of c ft/sec^2 results, determine the value of c necessary to ensure that an automobile traveling at 40 mi./hr. will stop 50 feet from the place where the brakes are applied.

By the method of Section 6-5(ii) the displacement s from the point where the brakes are applied is given by

$$s = v_0t - \frac{ct^2}{2} \tag{1}$$

where $v_0 = 40$ mph $= \frac{176}{3}$ ft./sec. and the velocity v by

$$v = v_0 - ct . \tag{2}$$

According to the conditions of the problem, we have $v = 0$ when $s = 50$, hence from (2)

$$t = \frac{v_0}{c}$$

and, putting this value of t in (1)

$$50 = \frac{1}{2}\,\frac{v_0^2}{c}$$

whence $c = \frac{v_0^2}{100} = \frac{1}{100}\left(\frac{176}{3}\right)^2 \approx 34.4$.

11. Show that

$$\int_a^b f(x)dx = \frac{b-a}{6}(f(a) + 4f(\frac{a+b}{2}) + f(b))$$

where f is any quadratic function.

Set $f(x) = Ax^2 + Bx + C$. It is clearly sufficient to prove the result for x^2 , x , and 1 and then take the appropriate linear combination:

$$\int_a^b dx = b - a ,$$

and

$$\frac{b-a}{6}[1 + 4 + 1] = b - a ;$$

$$\int_a^b x\, dx = \frac{1}{2}(b^2 - a^2)$$

and

$$\frac{b-a}{6}[a + 2(a + b) + b] = \frac{(b-a)}{6}[3(a + b)]$$

$$= \frac{1}{2}(b^2 - a^2) ;$$

$$\int_a^b x^2 dx = \frac{b^3 - a^3}{3}$$

and

$$\frac{b-a}{6}[a^2 + (a^2 + 2ab + b^2) + b^2] = \frac{1}{3}(b - a)[a^2 + ab + b^2]$$

$$= \frac{b^3 - a^3}{6} .$$

Note: This problem establishes that Simpson's rule of approximate integration is exact for quadratic polynomials. In fact, this property can be used as a requirement to derive Simpson's rule. It is curious then that Simpson's rule is also exact for cubics:

$$\int_a^b x^3\, dx = \frac{b^4 - a^4}{4}$$

and

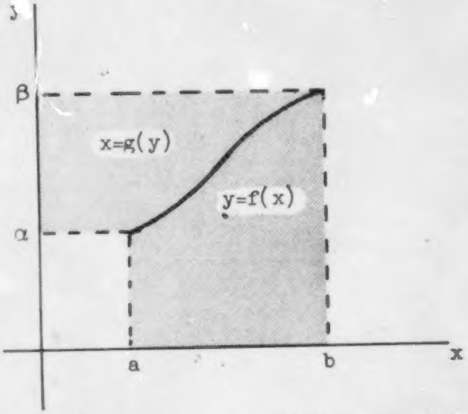
$$\frac{b-a}{6}[a^3 + \frac{1}{2}(a^3 + 3a^2b + 3ab^2 + b^3) + b^3] = \frac{b-a}{4}[a^3 + a^2b + ab^2 + b^3)]$$

$$= \frac{b^4 - a^4}{4} .$$

12. (a) Let f be continuous and increasing on $[a,b]$; let g be its inverse; and set $\alpha = f(a)$, $\beta = f(b)$. Prove

$$\int_{\alpha}^{\beta} g(y)dy = \beta b - \alpha a - \int_{a}^{b} f(x)dx .$$

Interpret this result geometrically when a and α are positive. (Hint: Compare with Exercises 6-2, Nos. 2-4.)



In the figure, it is clear that the sum of the shaded areas is

$$\int_{\alpha}^{\beta} g(y)dy + \int_{a}^{b} f(x)dx = \beta b - \alpha a .$$

To prove the result, use the idea of the solution of Exercises 6-2, No. 4. A partition $\sigma = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ corresponds to a partition $\sigma^* = \{f(x_0), f(x_1), \ldots, f(x_n)\}$ of $[\alpha,\beta]$. A term $f(x_k)(x_k - x_{k-1})$ of the upper Riemann sum U for f corresponds to a term $gf(x_{k-1})[f(x_k) - f(x_{k-1})]$ of the lower Riemann sum L^* for g . The sum of the two terms is

$$f(x_k)(x_k - x_{k-1}) + x_{k-1}[f(x_k) - f(x_{k-1})] = f(x_k)x_k - f(x_{k-1})x_{k-1} ;$$

hence, add the upper sum for f to the lower sum for g to obtain

$$U + L^* = \sum_{k=1}^{n} [f(x_k)x_k - f(x_{k-1})x_{k-1}] = f(x_n)x_n - f(x_0)x_0$$

$$= \beta b - \alpha a .$$

Now choose the partition σ so fine that

(a) $0 \leq U - \int_{a}^{b} f(x)dx < \epsilon$

and

$$\text{(b)}\quad 0 \leq \int_{\alpha}^{\beta} g(y)dy - L^* < \epsilon \ .$$

This can be done as follows. Take a partition $\tau_0 = \{u_0, u_1, \ldots, u_m\}$ of $[a,b]$ with $\nu(\tau_0) = \frac{\epsilon}{f(b) - f(a)}$ so that (a) is satisfied for the Riemann upper sum over τ. Similarly take a partition $\tau_0^{*} = \{v_0, v_1, \ldots, v_k\}$ of $[\alpha,\beta]$ with $\nu(\tau_0^{*}) = \frac{\epsilon}{(\beta) - g(\alpha)}$ so that (b) is satisfied for the lower Riemann sum over τ_0^*. Next, form the partition of $[a,b]$, $\tau_1 = \{g(v_0), g(v_1), \ldots, g(v_k)\}$ and adjoin the partition points of τ_1 to those of τ_0 to form σ ; i.e., $\sigma = \tau_0 \bigcup \tau_1$. With this choice of σ, both (a) and (b) are satisfied. Then, from (a) and (b),

$$-\epsilon < \int_a^b f(x)dx + \int_{\alpha}^{\beta} g(y)dy - (U + L^*) < \epsilon \ ,$$

and, since $U + L^* = \beta b - \alpha a$, the result follows.

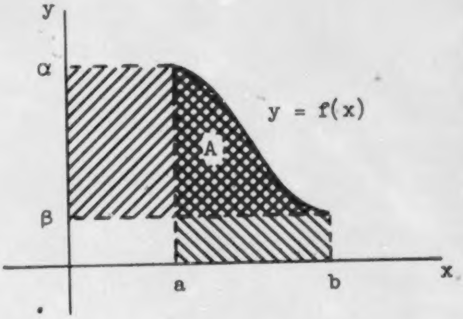
(b) Prove the formula and find a similar interpretation when f is decreasing on $[a,b]$.

The area of the region A indicated in the figure is given by

$$A = \int_a^b f(x)dx - \beta(b - a)$$

$$= \int_{\beta}^{\alpha} g(y)dy - a(\alpha - \beta)$$

from which the result follows immediately.



To prove the result formally, take for each term $f(x_k)(x_k - x_{k-1})$ of the Riemann lower sum L for f, the corresponding term $gf(x_{k-1})[f(x_{k-1}) - f(x_k)]$ of the lower sum L^* for g over $[\beta,\alpha]$. Since $gf(x_{k-1}) = x_{k-1}$,

$$L - L^* = \sum[f(x_k)x_k - f(x_{k-1})x_{k-1}] = \beta b - \alpha a .$$

As in Part (a) choose σ so that

$$0 \le \int_a^b f(x)dx - L < \epsilon$$

and

$$0 \le \int_\beta^\alpha g(y)dy - L^* < \epsilon$$

(Note the reversal of the order of the ends of integration).

Use $\int_\beta^\alpha = - \int_\alpha^\beta$ to obtain the result as before.

13. Check the arc length formula by applying it to a segment of a straight line and comparing with the formula for the distance between two points on a straight line.

The arclength for $y = mx + b$ from (x_1,y_1) to (x_2,y_2) is

$$\int_{x_1}^{x_2} \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \int_{x_1}^{x_2} \sqrt{1 + m^2}\, dx = \sqrt{1 + m^2}\, x \Big]_{x_1}^{x_2}$$

$$= \sqrt{1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2}\,(x_2 - x_1)$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} .$$

(More interesting arc length problems are postponed until greater technical facility in integration is developed.)

Teacher's Commentary

Chapter 7

BASIC INTEGRAL THEOREMS

The main purpose of this chapter is to prove the Fundamental Theorem of the Calculus. Continuity on an interval is a stronger condition than needed for either of the two propositions which make up the Fundamental Theorem, and more general results are proved in the text, but with this condition the reciprocal relation between the derivative and integral is established. Unfortunately, a general proof of integrability for continuous functions seems to require some subtlety. The proof given in Appendix 7 requires the concept of uniform continuity which is introduced without being named.

After Section 7-3 the word "integral" appears in context in several different senses: the integral of f, a specific "definite" integral of f over $[a,b]$: the integral of f, a specific function $F : x \longrightarrow \int_a^x f(x)dx$, later, more generally, a specific antiderivative: the indefinite integral of f, the class of antiderivatives of f, $\{F : F' = f\}$, written as $\int f(x)dx$ with no ends of integration, or as $F_0(x) + C$ where F_0 is any given member of the class: an integral of f, always an antiderivative.

Solutions Exercises 7-1

1. Show that any linear combination of monotone functions $\sum_{i=1}^{n} c_i f_i$ can be written as a sum of two functions $g + h$ where g is weakly increasing and h is weakly decreasing.

   If f is monotone then so is the function cf. If $c \geq 0$ then cf is monotone in the same sense as f ; if $c < 0$, then cf is monotone in the opposite sense. Furthermore if $f_1, f_2, \ldots, f_n$ are all monotone in the same sense then $f_1 + f_2 + \ldots + f_n$ is monotone in that sense.

   Any linear combination of monotone functions, $f = \sum_{i=1}^{n} c_i f_i$, is a sum of monotone functions $\sum_{i=1}^{n} \phi_i$ where $\phi_i = c_i f_i$. Divide the functions ϕ_i into two classes, those functions g_j which are weakly increasing (g_j includes the constant functions among the ϕ_i) and the remaining functions h_k. The functions h_k are monotone but not weakly increasing; therefore they must be weakly decreasing. Set

$$f = \sum_{i=1}^{n} \phi_i = \sum_{j=1}^{p} g_j + \sum_{k=1}^{q} h_k ,$$

where $p + q = n$. Put $g = \sum_{j=1}^{p} g_j$ and $h = \sum_{k=1}^{q} h_k$ to obtain the desired result.

2. For each of the following express f as the sum of monotone functions g and h and give formulas for $g(x)$ and $h(x)$ in each of the subintervals where f is monotone.

   Clearly, the answer is not unique; given any solution another can be found by adding, subtracting it from h.

   (a) $f : x \longrightarrow \arcsin(\sin x)$, $-\pi \leq x \leq \pi$ (Exercises 4-5, No. 1(a)).

   Recall that $\arcsin u$ has the domain $[-1,1]$ and the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Consequently,

$$f(x) = \begin{cases} -\pi - x , & -\pi \leq x \leq -\frac{\pi}{2} , \\ x , & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} , \\ \pi - x , & \frac{\pi}{2} \leq x \leq \pi . \end{cases}$$

   Take

$$g(x) = \begin{cases} 0 , & -\pi \leq x \leq -\frac{\pi}{2} , \\ x + \frac{\pi}{2} , & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} , \\ \pi , & \frac{\pi}{2} \leq x \leq \pi . \end{cases}$$

$$h(x) = \begin{cases} -\pi - x , & -\pi \leq x \leq -\frac{\pi}{2} , \\ -\frac{\pi}{2} , & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} , \\ -x , & \frac{\pi}{2} \leq x \leq \pi . \end{cases}$$

(b) $f : x \longrightarrow \arcsin(\cos x)$, $-\pi \le x \le \pi$ (See Exercises 4-5, No. 1(c)).

Here

$$f(x) = \begin{cases} \frac{\pi}{2} + x \,, & -\pi \le x \le 0 \,, \\ \frac{\pi}{2} - x \,, & 0 \le x \le \pi \,. \end{cases}$$

Set

$$g(x) = \begin{cases} \frac{\pi}{2} + x \,, \\ \frac{\pi}{2} \,. \end{cases} \qquad h(x) = \begin{cases} 0 \,, & -\pi \le x \le 0 \,, \\ -x \,, & 0 \le x \le \pi \,. \end{cases}$$

(c) $f : x \longrightarrow 4x^5 + 5x^4 - 20x^3 - 50x^2 - 40x$ (Example 5-8a and Exercise 5-8, No. 1).

See the solution to Exercises 5-8, No. 1.

$$g(x) = \begin{cases} f(x) \,, \\ 11 \,, \\ f(x) + 243 \,, \end{cases} \qquad h(x) = \begin{cases} 0 & , x \le -1 \,, \\ f(x) - 11 & , -1 \le x \le 2 \,, \\ -243 & , 2 \le x \,. \end{cases}$$

(d) $f : x \longrightarrow -x\sqrt{3 - x^2}$ (Exercises 5-8, No. 5).

See the solution to Exercises 5-8, Number 5.

$$g(x) = \begin{cases} f(x) \,, \\ \frac{3}{2} \,, \\ f(x) + \frac{3}{4} \,, \end{cases} \qquad h(x) = \begin{cases} 0 & , x \le -\sqrt{\frac{3}{2}} \,, \\ f(x) - \frac{3}{2} & , -\sqrt{\frac{3}{2}} \le x \le \sqrt{\frac{3}{2}} \,, \\ -\frac{3}{4} & , \sqrt{\frac{3}{2}} \le x \end{cases}$$

(e) $f : x \longrightarrow x^{2/3}(x - 2)^2$ (Exercises 5-8, No. 7(b)).

See the solution to Exercise 5-8, Number 7(b).

$$g(x) = \begin{cases} 0 & , x \le 0 , \\ f(x) & , 0 \le x \le \frac{\sqrt[3]{2}}{2} , \\ \frac{9\sqrt[3]{2}}{8} & , \frac{\sqrt[3]{2}}{2} \le x \le 2 , \\ f(x) + \frac{9\sqrt[3]{2}}{8} & , 2 \le x . \end{cases}$$

$$h(x) = \begin{cases} f(x) & , x \le 0 , \\ 0 & , 0 \le x \le \frac{\sqrt[3]{2}}{2} , \\ f(x) - \frac{9\sqrt[3]{2}}{8} & , \frac{\sqrt[3]{2}}{2} \le x \le 2 , \\ -\frac{9\sqrt[3]{2}}{8} & , 2 \le x . \end{cases}$$

Solutions Exercises 7-2

1. Prove the inequality (6) for $h < 0$.

With the same restrictions given in the text, we have for $-\delta < h < 0$, instead of Equation (5),

$$[f(x) - \epsilon](-h) \le \int_{x+h}^{x} f(t)dt \le [f(x) + \epsilon](-h) ,$$

whence

$$f(x) - \epsilon \le -\frac{1}{h} \int_{x+h}^{x} f(t)dt \le f(x) + \epsilon ,$$

or, by Definition 6-4b,

$$f(x) - \epsilon \le \frac{1}{h} \int_{x}^{x+h} f(t)dt \le f(x) + \epsilon .$$

2. **Prove**

$$\frac{d}{dx}\int_x^a f(t)dt = -f(x) \ .$$

$$\frac{d}{dx}\int_x^a f(t)dt = -\frac{d}{dx}\int_a^x f(t)dt \text{ , by Definition 6-4b,}$$

$$= -f(x) \qquad \text{ , by Lemma 7-2.}$$

3. (a) **Differentiate**

$$\int_a^{g(x)} f(t)dt$$

under suitable restrictions on f and g . (Hint: Consider the integral as a composition of functions.)

Set $\qquad \phi(x) = Ig(x) = \displaystyle\int_a^{g(x)} f(t)dt \ .$

By the Chain Rule (Theorem 4-6)

$$\phi'(x) = I'g(x)\cdot g'(x) = fg(x)\cdot g'(x) \ .$$

Under the conditions of Lemma 7-2 it is sufficient to require that a and $g(x)$ lie in an interval where f is continuous.

(b) Differentiate

$$\int_{h(x)}^{g(x)} f(t)dt$$

under suitable restrictions.

Let us suppose that $g(x)$ and $h(x)$ lie in an interval where f is continuous and let a be a point of that interval. We have

$$\phi(x) = \int_{h(x)}^{g(x)} f(t)dt = \int_a^{g(x)} f(t)dt - \int_a^{h(x)} f(t)dt$$

and from Part a

$$\phi'(x) = fg(x)\cdot g'(x) - fh(x)\cdot h'(x) \ .$$

4. From the Law of the Mean and Lemma 7-2 derive the Mean Value Theorem (See Exercises 6-4, No. 20(a)); that is, prove if f is continuous on $[a,b]$, then

$$\int_a^b f(t)dt = f(u)(b - a)$$

for some u in the open interval (a,b).

Set $\phi(x) = \int_a^x f(t)dt$. Since ϕ is differentiable for all x in $[a,b]$ we have by the Law of the Mean

$$\begin{aligned}\phi(b) - \phi(a) &= \phi'(u)(b - a)\\ &= f(u)(b - a)\end{aligned}$$

where u is some point of the open interval (a,b).

TC7-3. The Fundamental Theorem.

The Fundamental Theorem of Calculus consists of two parts:

$$\frac{d}{dx}\int_a^x f(t)dt = f(x) \,, \tag{1}$$

$$\int_a^x F'(t)dt = F(x) - F(a) \,. \tag{2}$$

The specification $f = F'$ relates the two parts of the theorem, but we use the requirement that f be continuous on the interval $[a,x]$ to insure that (1) and (2) are simultaneously valid. However, the two parts of the theorem can be considered as separate results applying to more general classes of function, not quite the same classes for both. Thus in Section 6-5 we prove (2) under the sole assumption that F' is integrable (not necessarily continuous). In Section 7-2 we prove (1) under the assumption that f is integrable and continuous at the one point x (not necessarily or the whole interval). The purpose of this more general development is to emphasize the two parts of the theorem as separate propositions rather than to obtain the greatest possible generality with the available means.

Solutions Exercises 7-3

1. For each of the following differentiation formulas write the corresponding integration formula.

(a) $\frac{d}{dx}(x^3 - 3x) = 3x^2 - 3$

(b) $\frac{d}{dx}\left(\frac{1}{1 + x^2}\right) = - \frac{2x}{(1 + x^2)^2}$

(c) $\frac{d\sqrt{x}}{dx} = \frac{1}{2\sqrt{x}}$

(d) $\frac{d(\sin^2 x)}{dx} = \sin 2x$

(a) $\int (3x^2 - 3)dx = x^3 - 3x + C$

(b) $\int - \frac{2x}{(1 + x^2)^2} dx = \frac{1}{1 + x^2} + C$

(c) $\int \frac{1}{2\sqrt{x}} dx = \sqrt{x} + C$

(d) $\int \sin 2x \, dx = \sin^2 x + C$

2. Find the given derivative and then write the corresponding integration formula.

(a) $\frac{d}{dx} a \cos bx$

(b) $\frac{d}{dx} \tan x^2$

(c) $\frac{d}{dx} \arctan \sqrt{x}$

(d) $\frac{d}{dx} \sqrt{1 - \cos x}$

(a) $-ab \sin bx$

$- \int ab \sin bx \, dx = a \cos bx + C$

(b) $2x \sec^2 x^2$

$\int 2x \sec^2 x^2 \, dx = \tan x^2 + C$

(c) $\frac{1}{2\sqrt{x}\,(1 + x)}$

$\int \frac{dx}{2\sqrt{x}\,(1 + x)} = \arctan \sqrt{x} + C$

(d) $\dfrac{\sin x}{2(1 - \cos x)^{1/2}}$

$$\int \frac{\sin x}{2(1 - \cos x)^{1/2}}\,dx = \sqrt{1 - \cos x} + C$$

3. Verify the following integration formulas.

(a) $\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$

(b) $\int (a + bx)^r dx = \dfrac{(a + bx)^{r+1}}{b(r + 1)} + C\ ,\ r \neq -1$

(c) $\int \dfrac{ax}{\sqrt{ax^2 + b}}\,dx = \sqrt{ax^2 + b} + C$

(d) $2\int \sin^2 x \sin 2x\,dx = \sin^4 x + C$

(e) $\int \dfrac{1}{\sqrt{x - x^2}}\,dx = 2 \arcsin \sqrt{x} + C$

(a) $\dfrac{d}{dx}\dfrac{\tan ax}{a} = \sec^2 ax$

(b) $\dfrac{d}{dx}\dfrac{(a + bx)^{r+1}}{b(r + 1)} = \dfrac{b(r + 1)(a + bx)^r}{b(r + 1)} = (a + bx)^r$

(c) $\dfrac{d}{dx}\sqrt{ax^2 + b} = \dfrac{ax}{\sqrt{ax^2 + b}}$

(d)
$$\begin{aligned}\frac{d}{dx}\sin^4 x &= 4 \sin^3 x \cos x\\ &= 2 \sin^2 x\,(2 \sin x \cos x)\\ &= 2 \sin^2 x \sin 2x\end{aligned}$$

(e)
$$\begin{aligned}\frac{d}{dx}\,2 \arcsin \sqrt{x} &= \frac{2}{2\sqrt{x}\,\sqrt{1 - x}}\\ &= \frac{1}{\sqrt{x - x^2}}\end{aligned}$$

4. (a) We have

$$\frac{d}{dx}\frac{1}{1+x^2} = -\frac{2x}{(1+x^2)^2},$$

hence $\frac{1}{1+x^2}$ is an antiderivative of $-\frac{2x}{(1+x^2)^2}$. Prove that $\frac{1}{1+x^2}$, although it is an antiderivative of $-\frac{2x}{(1+x^2)^2}$, is not an integral $\int_a^x -\frac{2x}{(1+x^2)^2}\,dx$.

If there were some value a for which

$$\frac{1}{1+x^2} = \int_a^x f(x)dx ,$$

then for $x = a$ we would have

$$\frac{1}{1+a^2} = 0 ,$$

contradicting the fact that $\frac{1}{1+a^2} > 0$ for all values of a.

(b) Give a necessary and sufficient condition that an antiderivative F of a function f continuous on the domain $[a,b]$ be an integral.

It is necessary and sufficient that $F(u) = 0$ for some value u in $[a,b]$. Then, from

$$F(x) = \int_u^x f(t)dt + C$$

we see on setting $x = u$ that $C = F(u) = 0$, whence

$$F(x) = \int_u^x f(t)dt . \tag{1}$$

Conversely, if there exists a value u in $[a,b]$ for which (1) is satisfied then $F(u) = 0$.

5. The initial value problem for the differential equation
$$DF = f$$
is to determine the function F when f is given and an "initial value" $F(a)$ is specified. Show under suitable conditions that there exists just one function F satisfying these conditions.

We are given the function f and seek a solution F of the differential equation

$$DF = f \tag{1}$$

satisfying an initial condition

$$F(a) = b \ . \tag{2}$$

Any solution of (1) is an anti-derivative, if f is continuous in an interval containing a, then any anti-derivative defined on that interval can be written in the form

$$F(x) = \int_a^x f(t)dt + C \ .$$

The initial condition (2) can only be satisfied if $b = F(a) = C$. In summary, if f is continuous on an interval containing a the only function satisfying the conditions (1) and (2) on the interval is

$$F(x) = b + \int_a^x f(t)dt \ .$$

6. Do Number 5 for the differential equation
$$xF'(x) + F(x) = f(x) \ .$$

Let $G(x) = xF(x)$. Since $G'(x) = xF'(x) + F(x)$, the differential equation reduces to

$$G'(x) = f(x) \tag{a}$$

which is the same type as that of Number 5. The only function satisfying the initial condition and the differential equation (a) is

$$G(x) = G(a) + \int_a^x f(t)dt \ .$$

Consequently,

$$F(x) = \frac{1}{x}\left[ab + \int_a^x f(t)dt\right] \ .$$

(The solution is meaningless, for $x = 0$, but then the equation does not involve F'.)

7. If f has an integrable derivative on $[a,b]$, prove that f can be represented as the sum of two monotone functions, $f = g + h$. (Hint: Consider the integrals of $f'(t) \pm |f'(t)|$.)

Since $|f(x)|$ is integrable (Exercises 6-4, No. 22(a)) it follows directly on addition that $f = g + h$ where

$$g(x) = f(a) + \frac{1}{2}\int_a^x (f'(t) + |f'(t)|)dt$$

$$h(x) = \frac{1}{2}\int_a^x (f'(t) - |f'(t)|)dt$$

since

$$g'(x) = \frac{1}{2}(f'(x) + |f'(x)|) \geq 0$$

$$h'(x) = \frac{1}{2}(f'(x) - |f'(x)|) \leq 0 .$$

It follows that g is weakly increasing and h is weakly decreasing.

8. Use the Fundamental Theorem to derive the linearity of integration (Theorem 6-4c) from the linearity of differentiation (Theorem 4-2a).

If f and g are continuous on $[a,b]$ it follows that $\alpha f + \beta g$ is continuous and that all three are integrable over $[a,b]$. From

$$F(x) = \int_a^x f(t)dt \ , \ G(x) = \int_a^x g(t)dt$$

we have

$$\begin{aligned} D[\alpha F(x) + \beta G(x)] &= \alpha F'(x) + \beta G'(x) \\ &= \alpha f(x) + \beta g(x) \end{aligned}$$

from which it follows that

$$[\alpha F(b) + \beta G(b)] - [\alpha F(a) + \beta G(a)] = \int_a^b [\alpha f(x) + \beta g(x)]\, dx$$

or

$$\begin{aligned} \alpha[F(b) - F(a)] + \beta[G(b) - G(a)] &= \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx \\ &= \int_a^b [\alpha f(x) + \beta g(x)]dx . \end{aligned}$$

9. A horizontal translation of a graph in the plane $(x,y) \longrightarrow (x + c, y)$ amounts only to a lateral shift for the standard region under the graph (Figure 7-3).

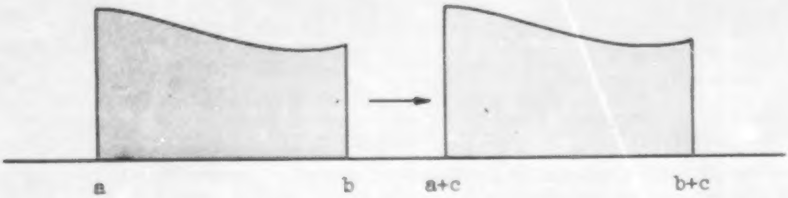


Figure 7-3

Verify that the expression of this geometrical property in terms of integrals is given by the formula,

$$\int_a^b f(x)dx = \int_{a+c}^{b+c} f(x - c)dx$$

(a) Derive the preceding formula using only the methods of Sections 6-1 to 6-4.

We write u instead of x in the second of the given integrals,

$$\int_{a+c}^{b+c} f(u - c)du .$$

Let $\sigma = \{x_0, x_1, \dots, x_n\}$ be any partition of $[a,b]$ and let $\tau = \{u_0, u_1, \dots, u_n\}$ be the corresponding partition of $[a + c, b + c]$ with $u_k = x_k + c$. If

$$u_{k-1} \leq u \leq u_k$$

then

$$u_{k-1} - c \leq u - c \leq u_k - c$$

or

$$x_{k-1} \leq u - c \leq x_k .$$

It follows that bounds on $f(x)$ on $[x_{k-1}, x_k]$ are bounds for $f(u - c)$ on $[u_{k-1}, u_k]$. Thus, for every upper sum for $f(x)$ over $[a,b]$

$$U_\sigma = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) ,$$

there is an equal upper sum for $f(u - c)$ over $[a + c , b + c]$,

$$U_\tau = \sum_{k=1}^{n} M_k(u_k - u_{k-1}) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) .$$

The same is true of the lower sums. We conclude that if either integral exists, then so does the other and that the two are equal.

(b) Derive the formula using the Fundamental Theorem.

Consider the integrals

$$\phi(t) = \int_a^t f(x)dx \; , \; \psi(t) = \int_{a+c}^{t+c} f(x - c)dx .$$

We have

$$\psi'(t) = f([t + c] - c) = f(t) = \phi'(t) .$$

It follows that

$$\phi(t) = \psi(t) + C$$

whence

$$\phi(b) - \phi(a) = \psi(b) - \psi(a)$$

or

$$\int_a^b f(x)dx = \int_{a+c}^{b+c} f(x - c)dx .$$

10. Equation (5) can be interpreted geometrically as stating that a uniform change of vertical scale by the factor λ , that is, a transformation $(x,y) \longrightarrow (x,\lambda y)$, multiplies areas by λ . Since there are no preferred directions in the plane--the choice of coordinate axes is only a useful convention--the same result must be true of a uniform change of scale in any direction. In particular, a scale transformation $(x,y) \longrightarrow (\lambda x,y)$ in the x direction must multiply areas by the factor λ . Verify that the expression of this geometrical property in terms of integrals is given by the formula

$$\int_a^b f(x)dx = \int_{a\lambda}^{b\lambda} \frac{1}{\lambda} f(\frac{x}{\lambda})dx , \qquad \lambda \neq 0 .$$

(a) Derive this formula using only the methods of Sections 6-1 to 6-4.

Let (u,v) denote coordinates in the transformed plane:

$$x \longrightarrow u = \lambda x \ , \ y \longrightarrow v = y \ .$$

The interval $[a,b]$ of the x-axis is mapped on the interval $[\lambda a, \lambda b]$ of the u-axis. (We assume $b > a$ and $\lambda > 0$. Otherwise details would have to be modified, but the general method is the same.) A point $(x,f(x))$ of the graph of f is mapped onto the point $(u,f(x)) = (u,f(\frac{u}{\lambda}))$. Since the mapping is presumed to multiply areas by the factor λ, we should have

$$\int_{a\lambda}^{b\lambda} f(\tfrac{u}{\lambda})\, du = \lambda \int_a^b f(x)dx \ .$$

(a) To each partition $\sigma = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ there corresponds the partition $\tau = \{u_0, u_1, \ldots, u_n\}$ where $u_k = \lambda x_k$. For

$x_{k-1} \le x \le x_k$ we have $u_{k-1} \le u \le u_k$ whence $x_{k-1} \le \frac{u}{\lambda} \le x_k$.
Thus $f(\frac{u}{\lambda})$ may be assigned the same bounds on $[u_{k-1}, u_k]$ as $f(x)$ on $[x_{k-1}, x_k]$. Furthermore, since $u_k - u_{k-1} = \lambda(x_k - x_{k-1})$, we have the same values for the upper and lower sums of $\lambda f(x)$ on $[a,b]$ as for $f(\frac{u}{\lambda})$ on $[\lambda a, \lambda b]$. We conclude that the two integrals are equal:

$$\lambda \int_a^b f(x)dx = \int_{a\lambda}^{b\lambda} f(\tfrac{x}{\lambda})dx \ , \qquad \lambda \ne 0 \ .$$

(b) Derive this formula using the Fundamental Theorem.

In treating Part (a) completely by the techniques of Sections 6-1 to 6-4 we would have to discuss all cases; we have only treated the case $\lambda > 0$ and $b > a$. Using the Fundamental Theorem we can immediately prove the result in all generality. Set

$$\phi(x) = \int_a^x f(t)dt \ , \ \psi(x) = \int_{\lambda a}^{\lambda x} \frac{1}{\lambda} f(\tfrac{t}{\lambda})dt \ .$$

We have

$$\begin{aligned} \psi'(x) &= \frac{1}{\lambda} f(\tfrac{\lambda x}{\lambda}) \cdot D(\lambda x) \\ &= f(x) = \phi'(x) \ . \end{aligned}$$

It follows that

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$$\psi(b) - \psi(a) = \phi(b) - \phi(a)$$

or

$$\int_a^b f(t)dt = \int_{\lambda a}^{\lambda b} \frac{1}{\lambda} f(\tfrac{t}{\lambda})dt \ .$$

(c) The number π is defined as the area of a circle of radius 1. Prove from the results of this exercise alone that the area of a circle of radius r is πr^2.

We have $\pi = \int_{-1}^{1} [f(x) - g(x)]dx$ where $f(x) = \sqrt{1 - x^2}$ and $g(x) = -\sqrt{1 - x^2}$, or

$$\pi = \int_{-1}^{1} 2\sqrt{1 - x^2}\, dx \ .$$

Consider the area of a circle of radius r,

$$\begin{aligned}
(1) \qquad A &= \int_{-r}^{r} 2\sqrt{r^2 - x^2}\, dx \\
&= \int_{-r}^{r} 2r\sqrt{1 - (\tfrac{x}{r})^2}\, dx \\
&= r^2 \int_{-r}^{r} \frac{2}{r} \sqrt{1 - (\tfrac{x}{r})^2}\, dx \\
&= r^2 \int_{-1}^{1} 2\sqrt{1 - x^2}\, dx \qquad \text{(from part (b))} \\
&= \pi r^2 \ .
\end{aligned}$$

(d) By the method of (c) find the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ .$$

For the given area we have

$$\int_{-a}^{a} 2b\sqrt{1 - \frac{x^2}{a^2}}\, dx = \frac{b}{a} \int_{-a}^{a} 2\sqrt{a^2 - x^2}\, dx \qquad \text{(Compare line (1) above.)}$$

$$= \frac{b}{a}(\pi a^2) = \pi ab \ .$$

11. Prove that

$$\int_a^b f(x)dx = \int_{(a+c)\lambda}^{(b+c)\lambda} \frac{1}{\lambda} f(\frac{y}{\lambda} - c)dy , \qquad \lambda \neq 0 .$$

This combines the results of Numbers 9 and 10:

$$\int_a^b f(x)dx = \int_{a+c}^{b+c} f(x - c)dx = \int_{(a+c)\lambda}^{(b+c)\lambda} \frac{1}{\lambda} f(\frac{x}{\lambda} - c)dx$$

where Number 9 and Number 10 are used in succession. Alternatively, apply the Fundamental Theorem as in the proofs of Numbers 9 and 10.

12. Using Number 11, show that

(a) $\int_0^\pi x\, f(\sin x)dx = \frac{\pi}{2} \int_0^\pi f(\sin x)dx$

(assume that the integrals exist).

From Number 11 with $a = 0$, $b = \pi$, $c = -\pi$ and $\lambda = -1$,

$$\int_0^\pi F(x)dx = \int_\pi^0 - F(\pi - x)dx$$

$$= \int_0^\pi F(\pi - x)dx .$$

Set $F(x) = xf(\sin x)$ to obtain

$$\int_0^\pi xf(\sin x)dx = \int_0^\pi (\pi - x)f(\sin(\pi - x))dx$$

$$= \int_0^\pi (\pi - x)f(\sin x)dx$$

$$= \pi \int_0^\pi f(\sin x)dx - \int_0^\pi x\, f(\sin x)dx ,$$

from which the result follows immediately.

(b) Evaluate $\int_0^\pi x \sin x\, dx$.

Apply the result of Part (a), to obtain

$$\int_0^x x \sin x \, dx = \frac{\pi}{2} \int_0^\pi \sin x$$

$$= \frac{\pi}{2}(-\cos x) \Big|_0^\pi$$

$$= \pi \, .$$

TC7-4. Formal Integration.

The purpose of this section is to make a token payment on the promise of the preceding section, that an integral calculus will be based on the differential calculus developed in Chapter 4. The bulk of the work on formal integration is in Chapter 10. The serious need for skill in technical integration is motivated by Chapter 9 and that is why we defer a full-scale treatment.

Solutions Exercises 7-4

1. Use knowledge of derivatives previously obtained to find an expression for each of the following.

(a) $\int \sin(ax + b)dx = -\frac{1}{a}\cos(ax + b) + C$

(b) $\int \sqrt{ax + b}\, dx = \frac{2}{3a}(ax + b)^{3/2} + C$

(c) $\int (5 + x)^{10}\, dx = \frac{(5 + x)^{11}}{11} + C$

(d) $\int \frac{1}{(a + x)^{2/3}}\, dx = 3(a + x)^{1/3} + C$

(e) $\int \frac{1}{\cos^2 ax}\, dx = \frac{\tan ax}{a} + C$

(f) $\int 2x \sin x^2\, dx = -\cos x^2 + C$

(g) $\int x^2(1 + 3x^3)^{2/3}\, dx = \frac{(1 + 3x^3)^{5/3}}{15} + C$

(h) $\int \frac{1}{\sqrt{1 - 4x^2}}\, dx = \frac{1}{2}\arcsin 2x + C$

(i)
$$\begin{aligned}\int \tan^2 bx\, dx &= \int (\sec^2 bx - 1)dx \\ &= \int \sec^2 bx\, dx - \int dx \\ &= \frac{\tan bx}{b} - x + C\end{aligned}$$

(j) $$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + C$$

(k) $$\int \sqrt{\frac{1-x}{1+x}}\,dx = \int \sqrt{\frac{1-x}{1+x}\cdot\frac{1-x}{1-x}}\,dx$$
$$= \int \frac{1-x}{\sqrt{1-x^2}}\,dx$$
$$= \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x}{\sqrt{1-x^2}}\,dx$$
$$= \arcsin x + (1-x^2)^{1/2} + C$$

(ℓ) $$\int \frac{2x}{1+x^4}\,dx = \int \frac{Dx^2}{1+(x^2)^2}\,dx$$
$$= \arctan x^2 + C$$

(m) $$\int \frac{x^2}{\sqrt{1-x^6}}\,dx = \frac{1}{3}\int \frac{3x^2}{\sqrt{1-(x^3)^2}}\,dx = \frac{\arcsin x^3}{3} + C$$

2. (a) $$\int_0^{\pi} |\cos t|\,dt = \int_0^{\pi/2} \cos t\,dt - \int_{\pi/2}^{\pi} \cos t\,dt$$
$$= \sin t\,\Big|_0^{\pi/2} - \sin t\,\Big|_{\pi/2}^{\pi} = 2$$

(b) $$\int_0^{1/2} \frac{1}{1+4x^2}\,dx = \frac{1}{2}\arctan 2x\,\Big|_0^{1/2} = \frac{\pi}{8}$$

(c) $$\int_{-\pi/4}^{0} \left(\frac{2}{\cos t}\right)^2 dt = 4\tan t\,\Big|_{-\pi/4}^{0} = 0 - (-4) = 4$$

(d) $$\int_0^{\pi} D_x \sin^2 x\,dx = \sin^2 x\,\Big|_0^{\pi} = 0$$

(e) $$\int_0^{\pi} x\sin^2 nx\,dx\ ,\ n \text{ an integer.}$$

Apply Exercises 7-3, Number 12 with $f(\sin x) = \sin^2 nx$:

$$\int_0^\pi x \sin^2 nx \, dx = \frac{\pi}{2}\int_0^\pi \sin^2 nx \, dx$$

$$= \frac{\pi}{2}\int_0^\pi \frac{1 - \cos 2nx}{2}\, dx$$

$$= \frac{\pi}{4}\left(x - \frac{\sin 2nx}{2n}\right)\Bigg|_0^{2\pi}$$

$$= \frac{\pi^2}{4} .$$

(To prove that $\sin^2 nx$ is a function of $\sin x$ on $[0,\pi]$ observe that on the subintervals $[0,\frac{\pi}{2}]$ and $[\frac{\pi}{2},\pi]$ the function $x \longrightarrow \sin x$ is strongly monotone. Consequently, on each of the subintervals the sine function has an inverse and there is a functional relation

$$\sin x \longrightarrow x \longrightarrow \sin^2 nx .$$

However, for a given point y in $[0,1]$ there are numbers x and $\pi - x$ in both subintervals such that $\sin x = \sin(\pi - x) = y$. To demonstrate the functional relation $\sin x \longrightarrow \sin^2 nx$ it is necessary to show that both x and $\pi - x$ yield the same value, $\sin^2 nx$. This is straightforward:

$$\sin^2 n(\pi - x) = [\sin n\pi \cos nx - \cos n\pi \sin nx]^2$$

$$= [(-1)^{n+1} \sin nx]^2 = \sin^2 nx .)$$

(f) $$\int_0^\pi x \cos^4 x \, dx = \int_0^\pi x\,(1 - \sin^2 x)^2 \, dx$$

$$= \frac{\pi}{2}\int_0^\pi (1 - \sin^2 x)^2 \qquad \text{(from Exercises 7-3, No. 12)}$$

$$= \frac{\pi}{2}\int_0^\pi \left(\frac{1 + \cos 2nx}{2}\right)^2$$

$$= \frac{\pi}{8}\int_0^\pi \left(1 + 2\cos 2nx + \frac{1 + \cos 4nx}{2}\right) dx$$

$$= \frac{3\pi^2}{16} .$$

3. (a) $$\int_{-1}^{1} x^{2/3}\,dx = \frac{3}{5}x^{5/3}\Big|_{-1}^{1} = \frac{6}{5}$$

(b) $$\int_{0}^{2} (x-1)^{2/3}\,dx = \frac{3}{5}(x-1)^{5/3}\Big|_{0}^{2} = \frac{6}{5}$$

(c) $$\int_{a}^{b} \frac{1}{\sqrt{1-x^2}}\,dx = \text{arcsin } b - \text{arcsin } a\,,\ \text{where } |a| < 1 \text{ and } |b| < 1\,.$$

(d) $\int_{a}^{b} \tan x \sec x\,dx = \sec b - \sec a$, where both a and b lie between successive zeros of $\cos x$, that is, in $[-\frac{\pi}{2}+n\pi\,, \frac{\pi}{2}+n\pi]$ for some integral value of n .

(e) $$\int_{1}^{2} (x^2+1)^{1/2}\,x\,dx = \frac{(x^2+1)^{3/2}}{3}\Big|_{1}^{2} = \frac{1}{3}(5^{3/2} - 2^{3/2})$$

$$= \frac{5\sqrt{5} - 2\sqrt{2}}{3}$$

(f) $$\int_{a}^{b} h'(gf(x)) \cdot g'f(x) \cdot f'(x)dx = hgf(b) - hgf(a)$$

(g) $$\int_{a}^{b} \sin x^3 \cos x^3\, x^2\,dx = \int_{a}^{b} \frac{\sin 2x^3}{2} \cdot \frac{D(2x^3)}{6}\,dx$$

$$= -\frac{1}{12}\cos 2x^3\Big|_{a}^{b}$$

$$= \frac{-\cos 2b^3 + \cos 2a^3}{12}$$

4. Consider the statement

$$\int_{0}^{1} (1-t^2)^{-1/2}dt = \frac{\pi}{2}\,.$$

By obtaining the corresponding indefinite integral, show how we might be led to believe this statement. Explain why the statement has no meaning in the present context and show how it can be given a meaning.

For $0 \leq 1 < 1$,

$$\int_{0}^{x} \frac{1}{\sqrt{1-t^2}}\,dt = \text{arcsin } x\,.$$

The function arc sine is defined on $[0,1]$, but the integrand is not defined at $t = 1$ and the integral from 0 to 1 is therefore meaningless. Nonetheless, the integral can be given a reasonable interpretation by the definition

$$\int_0^1 \frac{1}{\sqrt{1 - t^2}}\, dt = \lim_{x \to 1^-} \int_0^x \frac{1}{\sqrt{1 - t^2}}\, dt ,$$

i.e.,

$$\int_0^1 \frac{1}{\sqrt{1 - t^2}}\, dt = \lim_{x \to 1^-} \arcsin x = \arcsin 1 = \frac{\pi}{2} .$$

5. Find the lengths of the following curves between the indicated points.

(a) $y = \frac{2}{3}(x - 1)^{3/2}$, $(1,0)$ to $(10,18)$.

$$\int_1^{10} \sqrt{1 + (x - 1)}\, dx = \int_1^{10} x^{1/2}\, dx$$

$$= \frac{2}{3}(10)^{3/2} - \frac{2}{3}(1)^{3/2}$$

$$= \frac{2}{3}(10\sqrt{10} - 1) .$$

(b) $y = x^3 + \frac{1}{12x}$, $(1, \frac{13}{12})$ to $(2, \frac{193}{24})$

From $y' = 3x^2 - \frac{1}{12x^2}$, $1 + (y')^2 = (3x^2 + \frac{1}{12x^2})^2$, and the arclength is

$$\int_1^2 (3x^2 + \frac{1}{12x^2})dx = 2^3 - \frac{1}{12 \cdot 2} - (1^3 - \frac{1}{12 \cdot 1})$$

$$= 7\,\frac{1}{24} .$$

(c) $y = \frac{x^4}{4} + \frac{x^{-2}}{8}$, from a to b , where $0 < a < b$.

$f'(x) = x^3 - \frac{x^{-3}}{4}$ and $1 + f'(x)^2 = (x^3 + \frac{x^{-3}}{4})^2$. The arclength is

$$\int_a^b (x^3 + \frac{x^{-3}}{4})dx = \frac{b^4 - a^4}{4} - \frac{1}{8}(b^{-2} - a^{-2}) .$$

TC7-5. Estimates of Integrals.

In Example 7-5a we obtain estimates for the arctangent function which derive from integration of partial sums of the geometric series

$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 \dots + (-x^2)^n + \dots .$$

For $0 < |x| < 1$, the series is an alternative series of decreasing terms which have limit 0 . The error in curtailing the series then has a sign opposite to that of the first omitted term and is less than that term in absolute value. In the text we must use finite methods to obtain this result since the theory of series will not be developed until Chapter 14. In Exercises 7-5, Number 2 the result is obtained by direct comparison with a finite geometric progression.

In Example 7-5b we do something far more interesting. We hint at Taylor's Theorem (Chapter 13) by using bounds on a higher derivative of cosine or sine and knowledge of the intermediate derivatives to obtain estimates of the function. This thread is partially hidden because we appeal strongly to the special cyclic properties of sine and cosine under differentiation or integration, but the example will prepare for the understanding of the general pattern when it is presented in the proof of Taylor's Theorem.

Solutions Exercises 7-5

1. (a) Obtain good approximations to $\sin \frac{1}{5}$ and $\cos \frac{1}{10}$ and give a tolerance within which you are sure of the accuracy of these values.

From

$$1 - \frac{x^2}{2} \leq \cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24} ,$$

for $x = \frac{1}{10}$,

$$0.995 \leq \cos \frac{1}{10} \leq 0.995 + \frac{10^{-4}}{24} < 0.995 + 5 \cdot 10^{-6} .$$

To five decimal place accuracy,

$$\cos \frac{1}{10} = 0.99500 .$$

Take the value of $\sin \frac{1}{10}$ from the text to obtain

$$\begin{aligned} \sin \frac{1}{5} &= 2 \sin \frac{1}{10} \cos \frac{1}{10} \\ &= 2(0.099833 \pm \epsilon_1)(0.99500 \pm \epsilon_2) \end{aligned}$$

where $|\epsilon_1| < 5 \times 10^{-6}$, $|\epsilon_2| < 5 \times 10^{-5}$ (It is not usually worthwhile to carry places much beyond the smallest number of significant digits in any factor). This yields

$$\sin \frac{1}{5} = 2(0.099833)(0.99500) + \epsilon ,$$

with the estimate of error

$$\begin{aligned} |\epsilon| &\leq 2(0.1\ |\epsilon_2| + |\epsilon_1| + |\epsilon_1 \epsilon_2|) \\ &< 2(5 \times 10^{-6} + 5 \times 10^{-6} + 5 \times 10^{-11}) \\ &< 2 \times 10^{-5} + 10^{-10} . \end{aligned}$$

The approximation to $\sin \frac{1}{5}$ is then

$$\begin{aligned} \sin \frac{1}{5} &= 2(0.099833)(0.99500) \\ &= (0.099833)(2 - \frac{1}{100}) \\ &= 0.19867 + \epsilon_3 , \end{aligned}$$

and the round off error is

$$|\epsilon_3| < 2 \times 10^{-6} .$$

Finally,

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$$\sin \frac{1}{5} = 0.19867 \pm \epsilon$$

where $\epsilon < 2 \times 10^{-5} + 2 \times 10^{-6} + 10^{-10} < 3 \times 10^{-5}$. A more careful error estimate would show that, in fact, the figure for $\sin \frac{1}{5}$ is accurate to five decimal places.

(b) Provide a general formula for these estimates and prove by mathematical induction that the general formula is correct.

Estimate $\sin t$ and $\cos x$ from below and above by

$$(1) \qquad s_p(t) \leq \sin t \leq \sigma_q(t), \qquad 0 \leq t \leq \frac{\pi}{2} ,$$

and

$$(2) \qquad c_p(x) \leq \cos x \leq \gamma_q(x) , \qquad 0 \leq x \leq \frac{\pi}{2} ,$$

where

$$s_p(t) = \sum_{k=1}^{2p} \frac{(-1)^{k+1} x^{2k-1}}{(2k - 1)!} , \quad \sigma_q(t) = \sum_{k=1}^{2q-1} \frac{(-1)^{k+1} t^{2k-1}}{(2k - 1)!}$$

and

$$c_p(x) = \sum_{k=0}^{2p-1} \frac{(-1)^k x^{2k}}{(2k)!} , \; \gamma_q(x) = \sum_{k=0}^{2q} \frac{(-1)^k x^{2k}}{(2k)!} .$$

For $p = q = 1$, in particular

$$t - \frac{t^3}{3!} \leq \sin t \leq t$$

and

$$1 - \frac{x^2}{2!} \leq \cos x \leq 1 - \frac{x^2}{2!} + \frac{x^4}{4!} .$$

We shall show that if (2) holds for $p = q = r$ then both inequalities also hold for $p = q = r + 1$. In that way the inequalities are established for all positive integers p , q . From

$$\cos x \leq \gamma_r(x)$$

we have

$$\sin t \leq \int_0^t \gamma_r(x)dx$$

where $0 \leq t \leq \frac{\pi}{2}$. Integrating term-by-term, we obtain

$$\begin{aligned}\sin t &\leq \sum_{k=0}^{2r} \frac{(-1)^k t^{2k+1}}{(2k+1)!} \\ &\leq \sum_{j=1}^{2r+1} \frac{(-1)^{j-1} t^{2j-1}}{(2j-1)!} \\ &\leq \sigma_{r+1}(t) .\end{aligned}$$

In summary,

(a) If $\cos x \leq \gamma_r(x)$, then $\sin t \leq \sigma_{r+1}(t)$.

Similarly, it can be proved that

(b) If $c_r(x) < \cos x$, then $s_r(t) \leq \sin t$.

(c) If $s_r(t) \leq \sin t$, then $\cos x \leq \gamma_r(x)$.

(d) If $\sin t \leq \sigma_r(t)$, then $c_r(x) \leq \cos x$.

Thus, from (a) and (d) we have from $\cos x \leq \gamma_r(x)$ that

(e) $c_{r+1}(x) \leq \cos x$ and thence by (b)

(f) $s_{r+1}(t) \leq \sin t$ and thence by (c)

(g) $\cos x \leq \gamma_{r+1}(x)$.

Combining statements (a), (c), (f), (g) we have the desired result.

2. Evaluate.

(a) $\lim\limits_{x \to 0} \frac{6 \sin x - 6x + x^3}{x^5}$

Since

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!} ,$$

$$\frac{1}{20} - \frac{6x^2}{7!} < \frac{6 \sin x - 6x + x^3}{x^5} < \frac{1}{20} .$$

Apply the Squeeze Theorem to obtain the limit $\frac{1}{20}$.

(b) $\lim\limits_{x \to 0} \dfrac{\cos 2x - 1 + 2x^2}{x^4}$

$$1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} < \cos 2x < 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} ;$$

hence

$$\frac{2}{3} - \frac{2^6 x^2}{6!} < \frac{\cos 2x - 1 + 2x^2}{x^4} < \frac{2}{3} ,$$

and the limit is $\frac{2}{3}$.

(c) $\lim\limits_{x \to 0} \dfrac{\sin x - x}{x(1 - \cos x)}$

For $x > 0$, use

$$- \frac{x^3}{3!} < \sin x - x < - \frac{x^3}{3!} + \frac{x^5}{5!} ,$$

$$\frac{x^3}{2!} + \frac{x^5}{4!} < x - x \cos x < \frac{x^3}{2!} ,$$

and for $x < 0$ use the inequalities in reverse order. In either case, we obtain

$$\frac{- \frac{x^3}{3!}}{\frac{x^3}{2!}} < \frac{\sin x - x}{x - x \cos x} < \frac{- \frac{x^3}{3!} + \frac{x^5}{5!}}{\frac{x^3}{2!} - \frac{x^5}{4!}} ,$$

from which it follows that the limit is $- \frac{1}{3}$.

3. (a) Prove

$$1 - x^2 \le \frac{1}{1 + x^2} \le 1 - x^2 + x^4 .$$

We have

$$(1 - x^2)(1 + x^2) = 1 - x^4 \le 1$$

and

$$(1 - x^2 + x^4)(1 + x^2) = 1 + x^6 \ge 1$$

from which the inequality follows.

(b) **Prove, in general, that**

$$1 - x^2 + x^4 \dots (-x^2)^{2p-1} \le \frac{1}{1 + x^2} \le 1 - x^2 + x^4 + \dots + (-x^2)^{2q}$$

where p and q are any natural numbers.

The expressions on the left and right in this inequality are geometric progressions. We have

$$1 - x^2 + \dots + (-x^2)^{2p-1} = \frac{1 - (-x^2)^{2p}}{1 - (-x^2)} = \frac{1 - x^{4p}}{1 + x^2} \le \frac{1}{1 + x^2}$$

and

$$1 - x^2 + \dots + (-x^2)^{2q} = \frac{1 - (-x^2)^{2q+1}}{1 - (-x^2)} = \frac{1 + x^{4q+2}}{1 + x^2} \ge \frac{1}{1 + x^2} .$$

(c) Use the results of (a) and (b) to obtain upper and lower estimates for $\arctan x$.

Integrating term by term from 0 to t we have

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \dots - \frac{x^{4p-1}}{4p - 1} \le \arctan x$$

$$\le x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + \frac{x^{4q+1}}{4q + 1} .$$

(d) Obtain a better approximation to π than that of Example 7-5a and show that the approximation is accurate within a closer tolerance.

If $q = p = n$ in c , we have as the difference between the upper and lower estimates

$$\frac{x^{4n+1}}{4n + 1} .$$

In the text we accepted a tolerance of 0.4 in π for $x = \frac{1}{\sqrt{3}}$.

Now we accept a tolerance of 0.008 in $\frac{\pi}{6}$ (one decimal place accuracy in π). We wish to choose n so that

$$\frac{\left(\frac{1}{\sqrt{3}}\right)^{4n+1}}{4n + 1} < \frac{1}{125}$$

or

$$9^n \sqrt{3}\,(4n + 1) > 125 .$$

Clearly $n = 2$ satisfies this requirement, in fact we have

$$(9^2 \sqrt{3})9 > 125 \times 10 .$$

Thus, for $x = \frac{1}{\sqrt{3}}$,

$$\frac{\pi}{6} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \epsilon , \text{ where } 0 < \epsilon < 0.0008 .$$

We have

$$\frac{1}{\sqrt{3}} - \frac{1}{3(\sqrt{3})^3} + \frac{1}{5(\sqrt{3})^5} - \frac{1}{7(\sqrt{3})^7} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{9} + \frac{1}{5 \cdot 9} - \frac{1}{7 \cdot 27}\right)$$

and

$$\frac{1}{9} = 0.11111 \pm 1.2 \times 10^{-6}$$

$$\frac{1}{5 \cdot 9} = 0.02222 \pm 2.3 \times 10^{-6}$$

$$\frac{1}{7 \cdot 27} = 0.00529 \pm 1.1 \times 10^{-6} .$$

Consequently

$$\pi = 2\sqrt{3}\,(0.9058 \pm 2.5 \times 10^{-5}) \pm 6\epsilon .$$

Taking $\sqrt{3} = 1.732 \pm 5 \times 10^{-4}$ we have

$$\pi = (3.464)(0.9058) \pm 4.9 \times 10^{-3}$$

$$= 3.138 \pm 5.3 \times 10^{-3} .$$

Solutions Miscellaneous Exercises

1. Determine a number k for which each of the following integration formulas is correct

(a) $\int (5x + 2)^7 \, dx = k(5x + 2)^8 + C .$ $\qquad k = \frac{1}{40}$

(b) $\int k \sqrt{3x - 2} \, dx = (3x - 2)^{3/2} + C .$ $\qquad k = \frac{9}{2}$

(c) $\int \sin ax \, dx = k \cos ax + C .$ $\qquad k = -\frac{1}{a}$

(d) $\int \cos 2x \, dx = k \sin x \cos x + C .$ $\qquad k = 1$

2. Compute each of the following integrals.

(a) $$\int_0^{\pi/4} \sec^2 x \, dx = \tan x \Big|_0^{\pi/4} = 1$$

(b) $$\int_0^{\pi/2} \sin x \cos x \, dx = \frac{1}{2} \sin^2 x \Big|_0^{\pi/2} = \frac{1}{2}$$

(c) $$\int_0^{3\pi/2} |\sin x| dx = -\cos x \Big|_0^{\pi} + \cos x \Big|_{\pi}^{3\pi/2} = 3$$

(d) $$\int_0^{\pi/2} \sqrt{1 + \sin 2x} \, dx = \int_0^{\pi/2} \sqrt{\sin^2 x + 2 \sin x \cos x + \cos^2 x} \, dx$$

$$= \int_0^{\pi/2} (\sin x + \cos x) dx = -\cos x + \sin x \Big|_0^{\pi/2} = 2 .$$

(e) $$\int_0^{\pi/2} \cos^4 x \, dx = \int_0^{\pi/2} \left(\frac{1 + \cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4} \int_0^{\pi/2} (1 + 2 \cos 2x + \cos^2 2x) dx$$

$$= \frac{1}{4} \int_0^{\pi/2} \left(\frac{3}{2} + 2 \cos 2x + \frac{\cos 4x}{2}\right) dx = \frac{3\pi}{16} .$$

(f) $$\int_0^{\pi} \sin^{100} x \cos^{99} x \, dx$$

Apply the result of Exercises 7-3, Number 9 with $c = -\frac{\pi}{2}$ to obtain

$$\int_0^{\pi} \sin^{100} x \cos^{99} x \, dx = \int_{-\pi/2}^{\pi/2} [\sin(x + \tfrac{\pi}{2})]^{100} [\cos(x + \tfrac{\pi}{2})]^{99} dx$$

$$= - \int_{-\pi/2}^{\pi/2} \cos^{100} x \sin^{99} x \, dx$$

$$= 0 ,$$

since the integrand is an odd function (Exercises 6-4, No. 4).

(g) $$\int_0^{\pi/4} \sin^4 x \cos^3 x \, dx = \int_0^{\pi/4} \sin^4 x(1 - \sin^2 x)dx$$

$$= \int_0^{\pi/4} (\sin^4 x - \sin^6 x)D \sin x \, dx$$

$$= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} \Bigg|_0^{\pi/4}$$

$$= \frac{1}{20\sqrt{2}} - \frac{1}{56\sqrt{2}} = \frac{9}{280\sqrt{2}} .$$

(h) $$\int_0^{\pi/2} x \sin x \cos x \, dx = \int_0^{\pi/2} \frac{x \sin 2x}{2} dx$$

Apply Exercises 7-3, Number 10 with $\lambda = 2$ and Exercises 7-3, Number 12, to obtain

$$\frac{1}{2} \int_0^{\pi} \frac{1}{2}\left(\frac{x}{2}\right)\sin 2\left(\frac{x}{2}\right)dx = \frac{1}{8} \int_0^{\pi} x \sin x \, dx$$

$$= \frac{\pi}{16} \int_0^{\pi} \sin x \, dx$$

$$= \frac{\pi}{5} .$$

3. Find the area of the given standard region.

(a) $f : x \longrightarrow \sqrt{2x + 1}$ over $[0,2]$.

$$\int_0^2 \sqrt{2x + 1} \, dx = \frac{1}{3}(2x + 1)^{3/2} \Bigg|_0^2$$

$$= \frac{1}{3}[5^{3/2} - 1]$$

$$= \frac{1}{3}[5\sqrt{5} - 1]$$

(b) $f : x \longrightarrow \dfrac{1}{\sqrt{2x + 1}}$ over $[0,2]$ . .

$$\int_0^2 \frac{1}{\sqrt{2x + 1}} dx = \sqrt{2x + 1} \Bigg|_0^2$$

$$= \sqrt{5} - 1$$

(c) $f : x \longrightarrow x\sqrt{2x^2 + 1}$ over $[0,2]$.

$$\int_0^2 x\sqrt{2x^2 + 1}\,dx = \frac{(2x^2 + 1)^{3/2}}{6}\Bigg|_0^2$$

$$= \frac{1}{6}[27 - 1] = \frac{13}{3}$$

(d) $f : x \longrightarrow \dfrac{x}{\sqrt{2x + 1}}$ over $[0,2]$.

Use

$$\frac{x}{\sqrt{2x + 1}} = \frac{\frac{1}{2}(2x + 1) - \frac{1}{2}}{\sqrt{2x + 1}} = \frac{1}{2}\sqrt{2x + 1} - \frac{1}{2} \cdot \frac{1}{\sqrt{2x + 1}} .$$

From (a) and (b), then

$$\int_0^2 \frac{x}{\sqrt{2x + 1}}\,dx = \frac{1}{6}[5\sqrt{5} - 1] - \frac{1}{2}[\sqrt{5} - 1]$$

$$= \frac{\sqrt{5} + 1}{3} .$$

4. Compute each of the following integrals.

(a) $$\int_0^{\pi/2} (\sin x - 2\cos x)dx = -(\cos x + 2\sin x)\Bigg|_0^{\pi/2} = -1$$

(b) $$\int_{1/2}^1 (2x + \csc^2 x)dx = (x^2 - \cot x)\Bigg|_{1/2}^1 = \frac{3}{4} - \cot 1 + \cot\frac{1}{2}$$

(c) $$\int_0^{\pi/2} \sin(x + n\pi)dx ,$$ (n an integer)

$$= \int_0^{\pi/2} (\sin x\cos n\pi + \cos x\sin n\pi)dx$$

$$= \int_0^{\pi/2} (-1)^n \sin x\,dx = (-1)^{n+1}\cos x\Bigg|_0^{\pi/2}$$

$$= (-1)^n .$$

Alternatively,

$$\int_0^{\pi/2} \sin(x + n\pi)dx = -\cos(x + n\pi)\Bigg|_0^{\pi/2} = \cos n\pi = (-1)^n$$

(d) $$\int_0^{\pi} \sqrt{\frac{1 + \cos 2x}{2}}\, dx = \int_0^{\pi} |\cos x|\, dx = \int_0^{\pi/2} \cos x\, dx - \int_{\pi/2}^{\pi} \cos x\, dx$$

$$= 2 .$$

(e) $$\int_0^{100\pi} (2 \sin x - \sin 2x)^{99}\, dx$$

$$= 50 \int_0^{2\pi} (2 \sin x - \sin 2x)^{99}\, dx$$

(by Exercises 6-4, No. 12)

$$= 50 \int_{-\pi}^{\pi} [2 \sin (x + \pi) - \sin 2(x + \pi)]^{99} dx$$

(Exercises 7-3. No. 9 with $c = -\pi$)

$$= -50 \int_{-\pi}^{\pi} (2 \sin x + \sin 2x)^{99}\, dx$$

$$= 0 ,$$

since the integrand is an odd function (Exercises 6-4, No. 4).

5. Find the area of the standard region $f : x \longrightarrow \sqrt{\frac{1 + \cos 2x}{2}}$ over $[0,2\pi]$.

Use $\sqrt{\frac{1 + \cos 2x}{2}} = |\cos x|$. The required area is

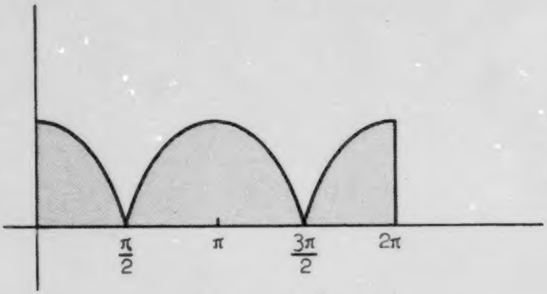
$$\int_0^{\pi/2} \cos x\, dx - \int_{\pi/2}^{3\pi/2} \cos x\, dx + \int_{3\pi/2}^{2\pi} \cos x\, dx$$

$$= \sin x \Big|_0^{\pi/2} - \sin x \Big|_{\pi/2}^{3\pi/2} + \sin x \Big|_{3\pi/2}^{2\pi}$$

$$= 4 .$$

Alternatively, from symmetry we find that the required area is

$$4 \int_0^{\pi/2} \cos x\, dx = 4 .$$



$f : x \longrightarrow |\cos x|$

6. **Solve each of the following differented equations subject to the prescribed conditions.**

(a) $\frac{dy}{dx} = x\sqrt{1 + x^2}$, $x = 0$, $y = -3$

$$y = \frac{1}{3}(1 + x^2)^{3/2} + C$$

$$-3 = \frac{1}{3} + C$$

$$y = \frac{1}{3}[(1 + x^2)^{3/2} - 10]$$

(b) $\frac{dv}{dt} = \sqrt[3]{2t + 1}$, $t = 0$, $v = v_0$

$$v = \frac{3}{8}(2t + 1)^{4/3} + C$$

$$v_0 = \frac{3}{8} + C$$

$$v = v_0 + \frac{3}{8}[(2t + 1)^{4/3} - 1]$$

(c) $\frac{ds}{dt} = \frac{3}{8}(2t + 1)^{4/3} + v_0 - \frac{3}{8}$, $t = 0$, $s = s_0$

$$s = s_0 + (v_0 - \frac{3}{8})t + \frac{9}{112}[(2t + 1)^{7/3} - 1]$$

7. Solve the differential equation

$$t^2 \frac{dy}{dt} + 2t\,y = \cos \pi t$$

subject to the condition $t = 1$, $y = 0$.

Put the equation in the form

$$D(t^2 y) = \cos \pi t \ ;$$

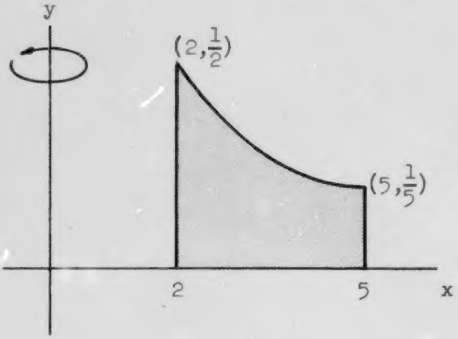
then,

$$t^2 y = \frac{1}{\pi} \sin \pi t + C \ .$$

Enter $t = 1$, $y = 0$ in this result to obtain $C = 0$ and

$$y = \frac{\sin \pi t}{\pi t^2} \ .$$

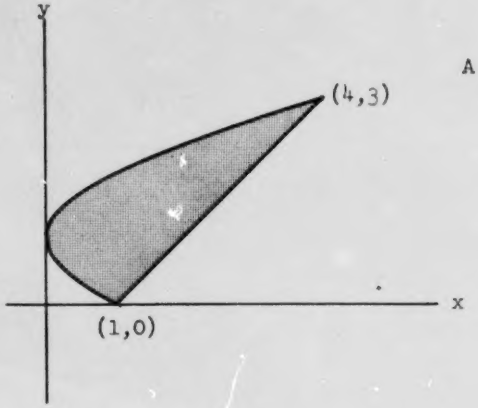
8. The standard region of $f : x \longrightarrow \frac{1}{x}$ over $[2,5]$ is rotated about the y-axis. Find the volume of the solid generated.



The desired volume is

$$\pi \int_0^{1/5} 25 dy + \pi \int_{1/5}^{1/2} \frac{1}{y^2}\, dy - \pi \int_0^{1/2} 4 dy = 5\pi - \frac{\pi}{y}\Bigg|_{1/5}^{1/2} - 2\pi = 6\pi \ .$$

9. Find the area of the region bounded by the parabola $x = (y - 1)^2$ and the line $y - x + 1 = 0$.



$$A = \int_0^3 [(y + 1) - (y - 1)^2]dy$$

$$= \int_0^3 (3y - y^2)dy = \frac{3y^2}{2} - \frac{y^3}{3} \Big|_0^3$$

$$= \frac{9}{2}$$

10. Find the area of the region bounded by the graph of $f : x \longrightarrow -x\sqrt{3 - x^2}$ and the x-axis. (See Exercises 5-8, No. 5.)

The area A of the region bounded by the curves is distinguished from the integral or signed area. Use the symmetry about the origin:

$$A = \int_{-\sqrt{3}}^{\sqrt{3}} |-x \sqrt{3 - x^2}\, |dx$$

$$= 2 \int_0^{\sqrt{3}} x \sqrt{3 - x^2}\, dx$$

$$= -\frac{2}{3} (3 - x^2)^{3/2} \Big|_0^{\sqrt{3}}$$

$$= 2\sqrt{3} .$$

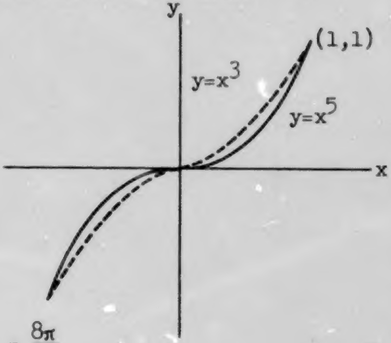
11. Find the area of the region bounded by the curves $y = x^5$ and $y = x^3$.

$$2 \int_0^1 (x^3 - x^5)dx = 2(\frac{x^4}{4} - \frac{x^6}{6}) \Big|_0^1$$

$$= \frac{1}{6}$$

(See Figure in No. 12.)

12. The region bounded by the curves $y = x^5$ and $y = x^3$ is rotated about the x-axis. Find the volume of the solid generated. Find the volume of the solid generated by rotating the region about the y-axis.



About the x-axis:

$$V = 2\pi \int_0^1 (x^6 - x^{10})dx$$

$$= 2\pi(\frac{1}{7} - \frac{1}{11}) = \frac{8\pi}{77}$$

About the y-axis:

$$V = 2\pi \int_0^1 (y^{2/5} - y^{2/3})dy = 2\pi(\frac{5}{7} - \frac{3}{5}) = \frac{8\pi}{35}$$

13. What is the mean value of the function $f : x \longrightarrow \sin x$ over the interval $[0,\frac{\pi}{2}]$? Of $g : x \longrightarrow \sin^2 x$ over $[0,\frac{\pi}{2}]$? (See Exercises 6-4, No. 20 b.)

$$\int_0^{\pi/2} \sin x \, dx = \frac{\pi}{2} \sin u = 1 \text{ , } 0 \leq u \leq \frac{\pi}{2} \text{ .}$$

$u = \arcsin \frac{2}{\pi}$. The mean value of f is $\frac{2}{\pi}$ over the interval $[0,\frac{\pi}{2}]$.

$$\int_0^{\pi/2} \sin^2 x \, dx = \frac{1}{2}(x - \frac{\sin 2x}{2})\Big|_0^{\pi/2} = \frac{\pi}{4} = \frac{\pi}{2} \sin^2 u \text{ , } (0 \leq u \leq \frac{\pi}{2})\text{; whence}$$

$\sin^2 u = \frac{1}{2}$. The mean value of g is $\frac{1}{2}$ on $[0,\frac{\pi}{2}]$.

14. Find the area bounded by the graphs of $f : x \longrightarrow \cos^2 x - \sin 2x + 1$ and $g : x \longrightarrow \cos 2x - \sin^2 x + 1$ between the lines $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$.

Use

$$f(x) = \cos^2 x - \sin 2x + 1$$
$$g(x) = \cos 2x - \sin^2 x + 1$$

to obtain

$$\phi(x) = f(x) - g(x) = 1 - \sin 2x - \cos 2x \text{ .}$$

Observe that $\phi(x) \geq 0$ for all x in the interval $[\frac{\pi}{4}, \frac{3\pi}{4}]$ since

$$\phi'(x) = -2 \cos 2x + 2 \sin 2x$$

is zero (in the given interval) only if $x = \frac{5\pi}{8}$,

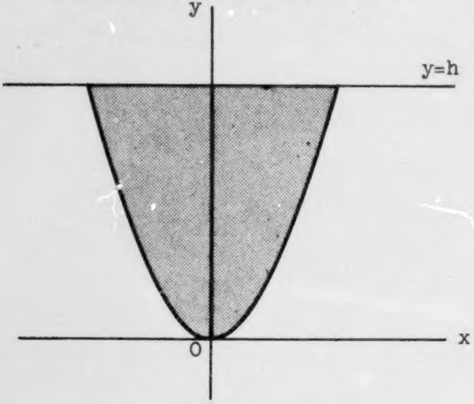
and $\phi(\frac{\pi}{4}) = 0$, $\phi(\frac{5\pi}{8}) = 1 + \sqrt{2}$, $\phi(\frac{3\pi}{4}) = 2$.

In other words $\phi(x)$ increases over the interval $[\frac{\pi}{4}, \frac{5\pi}{8}]$ and then decreases reaching the value 2 at $x = \frac{3\pi}{4}$. The desired area is

$$\int_{\pi/4}^{3\pi/4} [1 - \sin 2x - \cos 2x]dx = x + \frac{\cos 2x}{2} - \frac{\sin 2x}{2} \Big|_{\pi/4}^{3\pi/4}.$$

$$= 1 + \frac{\pi}{2}.$$

15. (a) Find the volume of the paraboloid generated by rotating the region in the first quadrant bounded by the parabola $y = x^2$, the y-axis, and the line $y = h$, $h > 0$.



$$v = \pi \int_0^h y\, dy$$

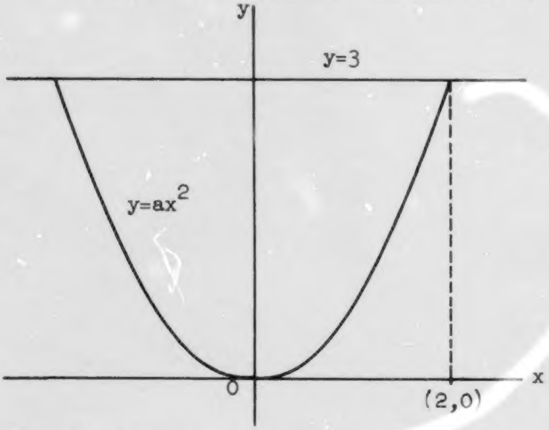
$$= \frac{\pi h^2}{2}$$

(b) Compare the volume of the paraboloid with the volume of a right circular cone of equal base-radius and height.

The cone has base radius $\sqrt{h}$, height h, and volume $\frac{\pi h^2}{3}$.

$V_{cone} = \frac{2}{3} V_{paraboloid}$.

16. Find the volume of a goblet in the form of a paraboloid of revolution which is 4 inches in diameter and 3 inches deep.



$$y = ax^2 \ , \ a = \frac{3}{4}$$

$$V = \pi \int_0^3 \frac{y}{a}\, dy$$

$$= 6\pi \text{ cubic inches}$$

17. Starting with

$$\frac{\pi}{4} = 7{,}854 \arctan \frac{1}{10{,}000} - \arctan \frac{1}{545{,}261}$$

(D. H. Lehmer), use Number 3c of Exercises 7-5 to find a rational approximation to π correct to at least 15 decimal places.

The result of Exercises 7-5, Number 3c is

$$x - \frac{x^3}{3} + \dots - \frac{x^{4p-1}}{4p-1} \le \arctan x \le x + \frac{x^3}{3} + \dots + \frac{x^{4q+1}}{4q+1} .$$

Since π is to be correct to 15 decimal places, set a tolerance of 1.25×10^{-16} in $\frac{\pi}{4}$. Take $p = q = n$ in the estimates for $\arctan x$; the difference between the upper and lower estimates is a bound on the error in taking either estimate for $\arctan x$. The maximum error in using Lehmer's formula is the sum of the errors for the two terms. Consequently take $n = s$ for the first term of the formula, and $n = t$ for the second, where

$$\epsilon_1 = \frac{7{,}854}{(10{,}000)^{4s+1}(4s+1)} < 6.25 \times 10^{-17}$$

$$\epsilon_2 = \frac{1}{(545{,}261)^{4t+1}(4t+1)} < 6.25 \times 10^{-17} .$$

For $s = 1$, we have $\epsilon_1 < \frac{10^4}{5 \times 10^{20}} \le 2 \times 10^{-17}$ and for $t = 1$, $\epsilon_2 < \frac{1}{(5 \times 10^5)^5 \times 5} < 10^{-25}$. Hence, to at least 15 decimal places

$$\pi \approx 4\left[7{,}854 \times \left(\frac{1}{10{,}000} - \frac{1}{3 \times 10{,}000^3}\right) - \frac{1}{545{,}261} + \frac{1}{3(545{,}261)^3}\right] .$$

18. **Evaluate.**

(a) $$\lim_{n\sim\infty}\left\{\frac{1}{\sqrt{n^2+n}}+\frac{1}{\sqrt{n^2+2n}}+\dots+\frac{1}{\sqrt{n^2+n\cdot n}}\right\}.$$

Express this as the limit of a Riemann sum and evaluate the integral.

$$\lim_{n\sim\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+kn}}=\lim_{n\sim\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{1+\frac{k}{n}}}\cdot\frac{1}{n}$$

$$=\int_0^1\frac{1}{\sqrt{1+x}}\,dx$$

$$=2\sqrt{1+x}\,\Big|_0^1=2\sqrt{2}-2\ .$$

(b) $$\lim_{n\sim\infty}\sum_{r=0}^{n}\frac{1}{\sqrt{n^2+r\sqrt{n}}}\ .$$

Note that $\frac{1}{\sqrt{n^2+n^{3/2}}}$ is the smallest term in the sum and that $\frac{1}{\sqrt{n^2}}$ is the largest. Hence

$$\frac{1}{\sqrt{1+\frac{1}{\sqrt{n}}}}\le\frac{n}{\sqrt{n^2+n^{3/2}}}<\sum_{r=0}^{n}\frac{1}{\sqrt{n^2+r\sqrt{n}}}<\frac{n+1}{\sqrt{n^2}}\le 1+\frac{1}{n}$$

By the Squeeze Theorem, then,

$$\lim_{r\sim\infty}\sum_{r=0}^{n}\frac{1}{\sqrt{n^2+r\sqrt{n}}}=1\ .$$

(c) $$\lim_{n\sim\infty}\frac{1}{n}\sum_{r=1}^{2n}\frac{r}{\sqrt{n^2+r^2}}\ .$$

Write as the limit of a Riemann sum and evaluate the integral:

$$\lim_{n\sim\infty}\sum_{r=1}^{2n}\frac{(\frac{r}{n})}{\sqrt{1+(\frac{r}{n})^2}}\cdot\frac{1}{n}=\int_0^2\frac{x}{\sqrt{1+x^2}}\,dx=\sqrt{1+x^2}\,\Big|_0^2=\sqrt{5}-1\ .$$

(d) $\lim_{n\to\infty} \left\{ \frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \dots + \frac{n}{n^2 + n^2} \right\}.$

Write as the limit of a Riemann sum and evaluate the integral:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1 + (\frac{k}{n})^2} \frac{1}{n} = \int_0^1 \frac{dx}{1 + x^2} = \arctan x \Big|_0^1 = \frac{\pi}{4} .$$

Teacher's Commentary
Chapter 8
LOGARITHMIC AND EXPONENTIAL FUNCTIONS

TC8-1. Introduction.

In this chapter we analyze a few special functions. These functions are not new to the student and their properties, not unfamiliar. The plan of attack is new. The method of developing the definition of logarithmic function is one often used by mathematicians: we start by examining the properties we want the function to have. In this case, the initial intuitive exploration involves the idea of powers, the exponential function $x \longrightarrow a^x$ (which is assumed to have the property $a^r a^s = a^{r+s}$ for all real numbers r, s, $a > 0$), its derivative, and inverse.

We have used a heuristic development. Another wonderful word of Greek origin to use by way of contrast is eristic, pertaining to argument which is intended to mislead.

At this time it might be worthwhile for the student to approximate $f'(0)$ for the function $f : x \longrightarrow 2^x$, using $2^{0.01} = 1.006955$, approximately. Later, he may compare his result with that of Exercises 8-2, Number 5(a), and anticipate that perhaps the constant of proportionality between $f(x)$ and $f'(x)$, namely $f'(0)$, is in fact a logarithm, $L(2)$. We note that any positive number may be expressed as a power of any other positive number, excluding 1. Using this fact we can approximate the value of the derivative at $x = 0$ for such functions as

$$g : x \longrightarrow \left(\tfrac{1}{2}\right)^x$$

$$h : x \longrightarrow 4^x ,$$

given only the approximation for $2^{0.01}$ (or more generally 2^h, for $|h|$ small). Thereafter, using (5) we can approximate $f'(x)$, $g'(x)$, and $h'(x)$ for other values of x.

Solutions Exercises 8-1

1. Prove that Properties (1) - (3) hold for rational exponents, $a > 0$.

Property 1. $a^m a^n = a^{m+n}$.

Take $m = \frac{p}{q}$ and $n = \frac{r}{s}$ where p, q, r, s are integers and $q > 0$, $s > 0$. Set $b = a^{p/q}$, $c = a^{r/s}$. Then

$$b^{sq} = a^{sp}, \quad c^{sq} = a^{qr};$$

hence, from Property 1 for integers,

$$b^{sq}c^{sq} = a^{sp}a^{qr}$$

implies

$$(bc)^{sq} = a^{sp+qr}.$$

But

$$(bc)^{sq} = (a^m a^n)^{sq}$$

and

$$a^{ps+qr} = a^{(\frac{p}{q}+\frac{r}{s})sq} = a^{(m+n)sq} = (a^{m+n})^{sq},$$

hence, $(a^m a^n)^{sq} = (a^{m+n})^{sq}$ from which the result follows by the one-to-one property of the mapping $x \longrightarrow x^{sq}$.

Property 2. $(a^m)^n = a^{mn}$.

Set $b = a^m$, $c = (a^m)^n$. With p, q, r, s as above, $a^p = b^q$ and $b^r = c^s$. Then

$$a^{pr} = b^{qr} = c^{sq},$$

whence

$$c = a^{pr/sq} \text{ or } (a^m)^n = a^{mn}.$$

Property 3. $(ab)^n = a^n b^n$.

Set $a^n = u$, $b^n = v$, $n = \frac{p}{q}$. Then $a^p = u^q$, $b^p = v^q$; hence,

$$(ab)^p = (uv)^q$$

and

$$(ab)^{p/q} = uv ,$$

which is the desired result.

2. Establish the monotone character and differentiability of the power functions $x \longrightarrow x^r$ $(x > 0 , r$ rational).

Set $r = \frac{p}{q}$ where p and q are integers and $q > 0$. Since $x^\alpha > 0$ for any exponent, the derivative

$$Dx^{p/q} = \frac{p}{q} x^{\frac{p}{q} - 1}$$

(Corollary to Theorem 4-3) has constant sign, the sign of ρ . If $p > 0$, $x^{p/q}$ is increasing; if $p < 0$, decreasing; if $p = 0$, constant, hence monotone in both senses.

3 Let f be defined for all real numbers and let f satisfy the functional equation

(1) $\qquad f(x + y) = f(x)f(y)$, for all x and y .

(a) Prove that if f is a solution of Equation (1) then either $f(0) = 0$ or $f(0) = 1$.

Set $x = y = 0$ in (1) to obtain

$$f(0) = f(0)^2 ;$$

hence, $f(0) = 0$ or $f(0) = 1$.

(b) Prove that if $f(0) \neq 0$, then there is no value of x for which $f(x) = 0$.

Set $y = -x$ in (1) to obtain

$$f(0) = f(x)f(-x) \neq 0 ,$$

whence $f(x) \neq 0$.

4. Let f satisfy the functional equation

(1) $$f(xy) = f(x) + f(y)$$

for all x, y in its domain.

(a) Prove that the function $f : x \longrightarrow 0$ is the only solution of (1) that is defined for all real numbers x.

If 0 is in the domain of f, then

$$f(0) = f(x) + f(0) ,$$

whence $f(x) = 0$ for all x in the domain of f. It follows that either $f : x \longrightarrow 0$ or else 0 is not in the domain of f.

(b) Prove that if f is a solution of (1) and the domain of f includes 1 and -1 but not 0, then $f(1) = 0$ and f is an even function.

From (1), $f(-1) = f(1) + f(-1)$; whence $f(1) = 0$. Now, take $x = y = -1$ in (1) to obtain $f(1) = 2f(-1) = 0$; whence $f(-1) = 0$. Consequently, $f(-1 \cdot x) = f(-1) + f(x)$ implies

$$f(-x) = f(x)$$

or that f is an even function.

(c) Prove that if f is a solution of (1) then $f(x^r) = rf(x)$, r rational.

First prove by induction that $f(x^n) = nf(x)$ for all natural numbers n. For $n = 1$, trivially $f(x^1) = 1 \cdot f(x)$. If $f(x^k) = kf(x)$ then

$$f(x^{k+1}) = f(x \cdot x^k) = f(x) + f(x^k) = (k + 1)f(x) .$$

It has already been shown for zero exponent that

$$f(x^0) = f(1) = 0 = 0 \cdot f(x) , \qquad (x \neq 0) .$$

For negative integral exponents and $x \neq 0$, we have

$$f(x^{-n}) = f\left(\left(\frac{1}{x}\right)^n\right) = n \cdot f\left(\frac{1}{x}\right) ;$$

but $f(\frac{1}{x}) = -f(x)$ since $f(x) + f(\frac{1}{x}) = f(x \cdot \frac{1}{x}) = 0$; therefore,

$$f(x^{-n}) = -nf(x) .$$

For rational exponents, $r = \frac{p}{q}$, where p and q are integers and $q > 0$,

$$f\left((x^r)^q\right) = f(x^p) ,$$

hence

$$q\, f(x^r) = pf(x)$$

which yields the desired result.

(d) Prove that if f is a solution of (1) and if f is differentiable at each $x \neq 0$, then $f'(x) = \frac{f'(1)}{x}$ for each $x \neq 0$.

For fixed x differentiate in (1) with respect to y to obtain $xf'(xy) = f'(y)$. Take $y = 1$ to obtain $f'(x) = \frac{f'(1)}{x}$, when $x \neq 0$.

Alternatively, calculate directly as follows:

$$\frac{f(x + h) - f(x)}{h} = \frac{f(x + h) + f(\frac{1}{x})}{h} = \frac{f(1 + \frac{h}{x})}{h} .$$

Now let $h = tx$; then

$$f'(x) = \lim_{h \sim 0} \frac{f(x + h) - f(x)}{h} = \frac{1}{x} \lim_{t \sim 0} \frac{f(1 + t)}{t}$$

$$= \frac{1}{x} \lim_{t \sim 0} \frac{f(1 + t) - f(1)}{t} = \frac{f'(1)}{x} .$$

(e) Using (d) show that any solution of (1) which is differentiable at each $x \neq 0$ is integrable on any closed interval $[a,b]$, where $ab > 0$.

From (d), $f'(x)$ has constant sign on any interval which does not include the origin. Thus f is monotone on $[a,b]$, hence integrable over $[a,b]$.

Solutions Exercises 8-2

1. Prove Corollaries 2 and 3 to Theorem 8-2.

Same as the solution of Exercises 8-1, Number 4(c).

2. (a) Show that the area A of the standard region $f : x \longrightarrow \frac{1}{x}$ over the interval $[1,2]$ satisfies the following inequality:

$$\frac{1}{3} + \frac{1}{4} < A < \frac{1}{2} + \frac{1}{3} .$$

Since f is decreasing on the interval $[1,2]$, obtain for the partition $\{1, \frac{3}{2}, 2\}$ the upper and lower sums

$$U = \frac{1}{2}\{f(1) + f(\tfrac{3}{2})\} = \frac{1}{2} + \frac{1}{3} ,$$

$$L = \frac{1}{2}\{f(\tfrac{3}{2}) + f(2)\} = \frac{1}{3} + \frac{1}{4} ,$$

so that $\frac{1}{3} + \frac{1}{4} < A < \frac{1}{2} + \frac{1}{3}$.

(b) Approximate A to the nearest $\frac{1}{10}$. (You need not carry out the calculation; i.e., represent A as a sum.)

For a subdivision of $[1,2]$ into n equal subintervals,

$$U - L = \frac{1}{n}[f(1) - f(2)] = \frac{1}{2n} .$$

Approximate A by $\frac{1}{2}(U + L)$. Then, to the required decimal accuracy

$$|A - \frac{1}{2}(U + L)| \leq \frac{1}{4n} < \frac{1}{20} .$$

Take $n = 6$; then

$$U = \frac{1}{6}\{f(1) + f(\tfrac{7}{6}) + f(\tfrac{8}{6}) + \ldots + f(\tfrac{11}{6})\}$$

$$L = \frac{1}{6}\{f(\tfrac{7}{6}) + f(\tfrac{8}{6}) + \ldots + f(\tfrac{11}{6}) + f(2)\}$$

and

$$A \approx \frac{1}{2}(U + L) = \frac{1}{2 \cdot 6} + \frac{1}{7} + \frac{1}{8} + \ldots + \frac{1}{11} + \frac{1}{2 \cdot 12} .$$

3. (a) For $a > 1$ show that the area A of the standard region of $f : x \longrightarrow \frac{1}{x}$ over $[1,a]$ satisfies the inequalities

$$1 - \frac{1}{a} < A < a - 1 .$$

On the interval $[1,a]$

(1) $$\frac{1}{a} \leq \frac{1}{x} \leq 1 .$$

Integrate to obtain

$$\frac{a - 1}{a} < A < a - 1$$

where strict inequality holds for the integrals since it holds in (1) for some x in the interval (Exercises 6-4, No. 18).

(b) For $0 < a < 1$ show that A satisfies

$$1 - a < A < \frac{1}{a} - 1$$

On the interval $[a,1]$,

$$1 \leq \frac{1}{x} \leq \frac{1}{a} .$$

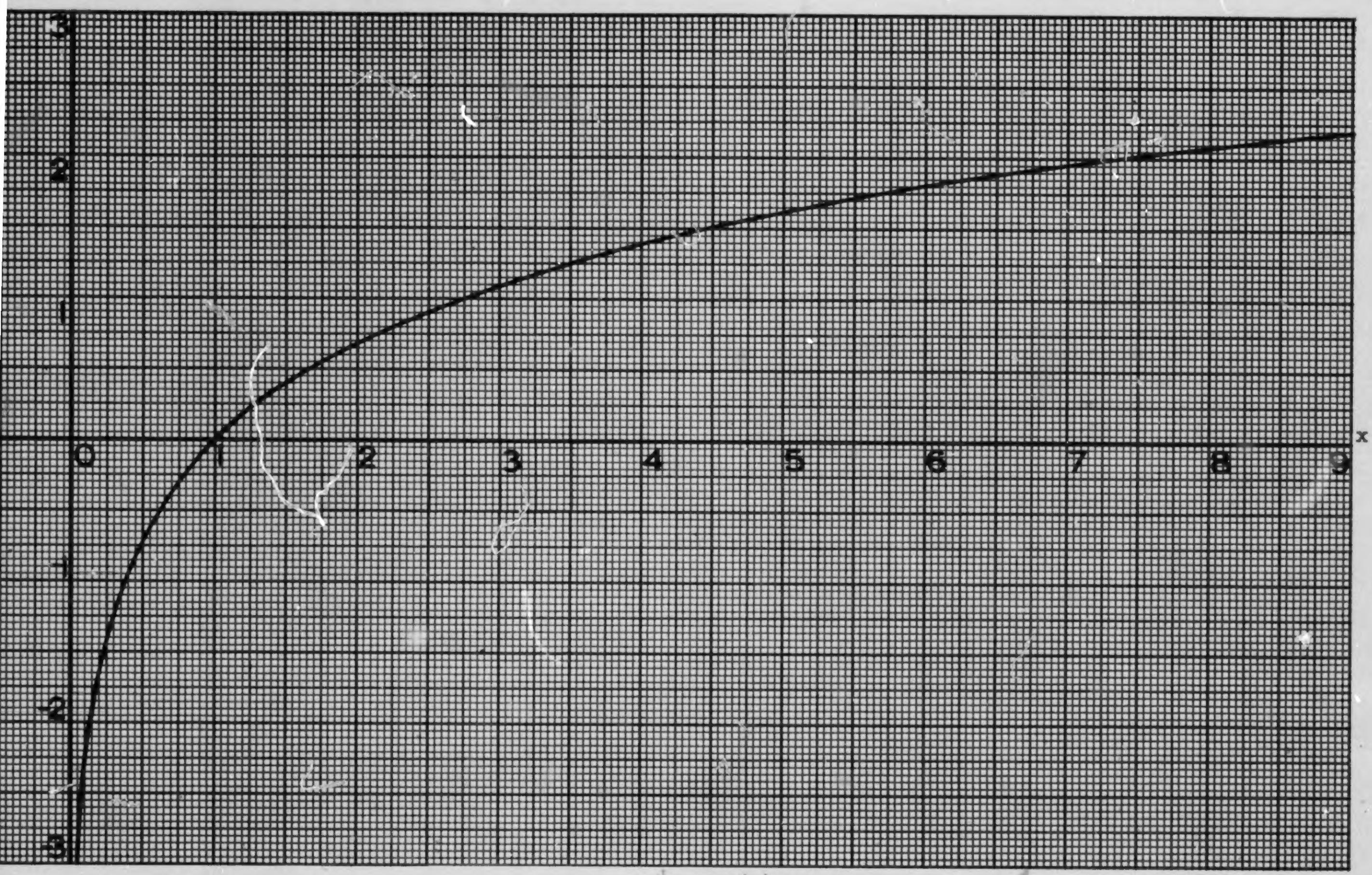
Integrate from a to 1 to obtain

$$1 - a < A < \frac{1 - a}{a} .$$

4. (a) Make a careful plot of the graph of L using $L(2) = 0.693$ and interpolating further values between those already given.

(b) Draw the tangent to the graph of L at each of the points $x = \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8$. Do these tangents conform to the plot in Part (a)?

(c) Obtain the solution of the equation $L(x) = 1$ with the accuracy available from your plot.



x ⟶ L(x)

5. (a) Using estimates by means of upper and lower sums, how many values of the integrand $\frac{1}{x}$ are needed to calculate $L(2)$ within the tolerance indicated by $L(2) = 0.693$?

$L(2)$ must be calculated with error less than $\frac{1}{2000}$. Since, for a subdivision of the interval $[1,2]$ into n equal parts, the difference between the upper and lower Riemann sums satisfies

$$U - L < \frac{1}{2n}$$

and since

$$L \leq L(2) \leq U ,$$

the mean of the upper and lower sums satisfies the error estimate

$$\left|L(2) - \frac{U + L}{2}\right| < \frac{1}{4n} .$$

Therefore, take $n = 500$. The value of $\frac{1}{x}$ must be obtained at the $n + 1 = 501$ partition points.

(b) Using the method of approximating $\frac{1}{t}$ by a linear function (see Section 5-3(ii) and 5-7) on each interval of a subdivision of $[1,2]$ show how to estimate $L(2)$ and determine the number of values of $\frac{1}{x}$ needed to give $L(2)$ accurately to three decimal places.

Take a subdivision of $[1,2]$ into n equal subintervals $[x_{k-1},x_k]$ where $x_k = 1 + \frac{k}{n}$, $(k = 0,1,2,\ldots,n)$. Let $g_k(x)$ represent the linear approximation to $\frac{1}{x}$ and suppose that

$$\left|g_k(x) - \frac{1}{x}\right| < e_k$$

on $[x_{k-1},x_k]$. Then

$$\left|L(2) - \sum_{k=1}^{n}\int_{x_{k-1}}^{x_k} g_k(x)dx\right| = \left|\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k} \left(g_k(x) - \frac{1}{x}\right)dx\right|$$

$$\leq \sum_{k=1}^{n}\left|\int_{x_{k-1}}^{x_k} \left(g_k(x) - \frac{1}{x}\right)dx\right| \leq \sum_{k=1}^{n}\int_{x_{k-1}}^{x_k} \left|g_k(x) - \frac{1}{x}\right|dx ,$$

which yields the upper error estimate $e = \sum_{k=1}^{n} e_k(x_k - x_{k-1})$.

The problem is to make $e < \frac{1}{2000}$.

First approximate the graph of f over $[x_{k-1},x_k]$ by the tangent line at x_{k-1} as in Section 5-7; i.e., take

$$g(x) = f(x_{k-1}) + f'(x_{k-1})(x - x_{k-1}) .$$

Then, from Section 5-7,

$$\begin{aligned} e_k &\le |f''(u_k)|(x_k - x_{k-1})^2 \qquad\qquad (x_{k-1} < u_k < x_k) \\ &\le \frac{1}{u_k^2}(x_k - x_{k-1})^2 \\ &\le \frac{1}{n^2} , \end{aligned}$$

since $u_k > 1$ and $x_k - x_{k-1} = \frac{1}{n}$. Consequently,

$$e = \sum_{k=1}^{n} e_k(x_k - x_{k-1}) \le \sum_{k=1}^{n} \frac{1}{n^2} \cdot \frac{1}{n} \le \frac{1}{n^2} .$$

To insure $e < \frac{1}{2000}$, take $n^2 > 2000$, say $n = 45$.

A point which should not be neglected, even in the era of large-scale digital computers, is that it often pays to do a little preliminary analysis before engaging in compution. A crude error estimate, such as obtained above, may lead to unnecessary extra computation.

The estimate is improved by making use of Equation (1), Section 5-7, and integrating: for x in $[x_{k-1},x_k]$,

$$\left|g_k(x) - \frac{1}{x}\right| \le \frac{1}{(x_{k-1})^2}(x - x_{k-1})^2 ;$$

whence

$$\int_{x_{k-1}}^{x_k} |g_k(x) - \tfrac{1}{x}|dx \le \frac{1}{(x_{k-1})^2}\int_{x_{k-1}}^{x_k} (x - x_{k-1})^2\,dx$$

$$\le \frac{1}{(x_{k-1})^2}\,\frac{(x - x_{k-1})^3}{3}\Bigg|_{x_{k-1}}^{x_k}$$

$$\le \frac{(x_k - x_{k-1})^3}{3(x_{k-1})^2}$$

$$\le \frac{1}{3n^3}\ .$$

Now sum from 1 to n to obtain the upper error estimate $\frac{1}{3n^2}$.

To make $\frac{1}{3n^2} < \frac{1}{2000}$, take $n = 26$.

There was another opportunity to get a better error estimate. Note that

$$\int_{x_{k-1}}^{x_k} |g_k(x) - \tfrac{1}{x}|dx \le \frac{1}{3n^3(x_{k-1})^2}$$

$$\le \frac{1}{3n^2}\left[\frac{1}{(x_{k-1})^2}\,(x_{k-1} - x_{k-2})\right]$$

$$\le \frac{1}{3n^2}\int_{x_{k-2}}^{x_{k-1}} \frac{1}{x^2}\,dx\ .$$

Now sum from 1 to n to get the upper error estimate

$$e \le \frac{1}{3n^2}\int_{1-\frac{1}{n}}^{2-\frac{1}{n}} \frac{1}{x^2}\,dx \le \frac{1}{3n^2}\left[\frac{1}{1-\frac{1}{n}} - \frac{1}{2-\frac{1}{n}}\right]$$

$$\le \frac{1}{3(n-1)(2n-1)} \le \frac{1}{3(n-1)(2n-2)}$$

$$\le \frac{1}{6(n-1)^2}\ .$$

To make $\frac{1}{6(n-1)^2} < \frac{1}{2000}$, take $n = 19$.

Thus the estimated length of computation has been cut by more than half, from $n = 45$ to $n = 19$. (In practice, a machine operator may let a machine run until it makes no further changes at the desired level of significance, but this may not be economical practice and it certainly is not foolproof.)

6. (a) Starting from

$$\frac{1}{1 - x} = 1 + x + x^2 + \dots + x^{m-1} + \frac{x^m}{1 - x}$$

show that

$$-L(1 - x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^m}{m} + R_m$$

where

$$R_m = \int_0^x \frac{y^m}{1 - y}\, dy .$$

Integrate from 0 to x .

(b) For the interval $0 \leq x < 1$, show that

$$0 \leq R_m \leq \frac{x^{m+1}}{(m + 1)(1 - x)}$$

and for $-1 \leq x \leq 0$ show that

$$|R_m| \leq \frac{|x|^{m+1}}{m + 1} .$$

For $0 \leq x < 1$

$$\begin{aligned} R_m = \int_0^x \frac{y^m}{1 - y}\, dy &< \int_0^x \frac{y^m}{1 - x}\, dx \\ &\leq \frac{1}{1 - x} \int_0^x y^m\, dy \\ &\leq \frac{x^{m+1}}{(m + 1)(1 - x)} . \end{aligned}$$

For $-1 < x < 0$, $1 - x > 1$ and

$$\begin{aligned} |R_m| = \left|(-1)^{m+1} \int_x^0 \frac{(-y)^m}{1 - y}\, dy\right| &< \int_x^0 \frac{(-y)^m}{1 - x}\, dy \\ &\leq \frac{(-1)^m}{1 - x} \frac{y^{n+1}}{m - 1} \Big|_x^0 \\ &\leq \frac{(-x)^{m+1}}{(1 - x)(m + 1)} \leq \frac{|x|^{m+1}}{(1 - x)(m + 1)} \leq \frac{|x|^{m+1}}{m + 1} . \end{aligned}$$

(c) Show that

$$\frac{1}{2} L\left(\frac{1 + x}{1 - x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{2n-1}}{2n - 1} + R_n^* ,$$

$$|x| < 1 \text{ , where } |R_n^*| \leq \frac{|x|^{2n}}{2n} \max \left(\frac{1}{1 - x}, 1\right) .$$

Combine (a) and (b) to obtain

$$-L(1 - x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^{2n-1}}{2n - 1} + R'_n$$

where $|R'_n| \leq \frac{|x|^{2n}}{2n} \max(\frac{1}{1 - |x|}, 1)$. Replace x by $-x$ in this result to obtain

$$L(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{x^{2n-1}}{2n - 1} + R''_n$$

where $|R''_n| \leq \frac{|x|^{2n}}{2n} \max(\frac{1}{1 - |x|}, 1)$, also. Now add and divide by 2 to obtain the desired result. (Note that for $|x| < 1$ $\max(\frac{1}{1 - |x|}, 1) = \max(\frac{1}{1 - x}, 1)$.)

(d) Using Part (c), obtain a value for $L(2)$ accurate to four decimal places.

Take $x = \frac{1}{3}$ in Part (c). The error tolerance is $\frac{1}{2} \times 10^{-4}$. Use the error estimate above to put an upper bound on the error in $L(2)$:

$$\begin{aligned} 2|R_n^*| &\leq \frac{2}{2n}(\frac{1}{3})^{2n} \max(\frac{3}{2}, 1) \\ &\leq \frac{1}{2n \cdot 3^{2n-1}} . \end{aligned}$$

To stay within the error tolerance, fix

$$\frac{1}{2n \cdot 3^{2n-1}} < \frac{1}{2 \times 10^4}$$

or

$$2n \cdot 3^{2n-1} > 2 \times 10^4 .$$

A reasonable try for n from the approximation $9^4 \approx 10^4$ is $n = 4$; for this value,

$$2n \cdot 3^{2n-1} = 8 \times 2{,}187 < 20{,}000 .$$

But if $n = 5$, then

$$2n \cdot 3^{2n-1} = 10 \times 19{,}683 > 20{,}000 .$$

To the required decimal accuracy

$$L(2) \approx 2[\frac{1}{3} + \frac{1}{3}(\frac{1}{3})^3 + \frac{1}{5}(\frac{1}{3})^5 + \frac{1}{7}(\frac{1}{3})^7 + \frac{1}{9}(\frac{1}{3})^9]$$

$$\approx .6932 .$$

(In obtaining the decimal representation the round-off error must also be controlled.)

(e) By setting $x = \frac{1}{2}$ in Part (c), we could obtain a value for $L(3)$. However, it is more efficient to calculate $L(3)$ by setting $\frac{1 + x}{1 - x} = \frac{3}{2}$ and using both Parts (c) and (d). Calculate $L(3)$ this way accurate to four decimal places.

For $\frac{1 + x}{1 - x} = \frac{3}{2}$, $x = \frac{1}{5}$ and

$$\frac{1}{2}L(\frac{3}{2}) = \frac{1}{5} + \frac{1}{3}(\frac{1}{5})^3 + \ldots + \frac{1}{2n - 1}(\frac{1}{5})^{2n-1} + R_n^*$$

where we impose

$$2|R_n^*| \leq \frac{2}{2n}(\frac{1}{5})^{2n} \frac{5}{4} < \frac{1}{2 \times 10^4}$$

or

$$n \cdot 5^{2n-1} > 5 \times 10^3$$

or

$$n \cdot 5^{2n-5} > 8$$

which is clearly first satisfied for $n = 3$. To four decimal place accuracy, then

$$L(\frac{3}{2}) = 2[\frac{1}{5} + \frac{1}{3}(\frac{1}{5})^3 + \frac{1}{5}(\frac{1}{5})^5] = .4055 .$$

Add the figure for $L(2)$ to obtain

$$L(3) = 1.0987 .$$

The last digit, however, may be off by 1 (in fact, it is), that is, by the sum of the allowed errors in the two figures for $L(\frac{3}{2})$ and $L(2)$.

(f) Obtain a value for $L(5)$ by setting

$$\frac{1 + x}{1 - x} = \frac{5}{4} .$$

Under the given condition $x = \frac{1}{9}$: Impose

$$2|R_n^*| \leq \frac{2}{2n}\left(\frac{1}{9}\right)^{2n}\frac{9}{8} < \frac{1}{2 \times 10^4}$$

or

$$n \cdot 9^{2n-1} > 2{,}500 \ .$$

This is satisfied first by $n = 3$. To four significant figures

$$L\left(\tfrac{5}{4}\right) = 2\left[\tfrac{1}{9} + \tfrac{1}{3}\left(\tfrac{1}{9}\right)^3 + \tfrac{1}{5}\left(\tfrac{1}{9}\right)^5\right] = .2232 \ .$$

Thus,

$$L(5) = L\left(\tfrac{5}{4}\right) + 2L(2) = 1.6096$$

where, as before, the last place is in doubt.

(g) *Give several ways of obtaining $L(11)$: Which is most efficient?*

$$\begin{aligned} L(11) &= L\left(\tfrac{11}{10}\right) + L(2) + L(5) \\ &= L\left(\tfrac{33}{32}\right) + 5L(2) - L(3) \\ &= \tfrac{1}{2}\left(L\left(\tfrac{121}{120}\right) + 3L(2) + L(3) + L(5)\right) \\ &= L\left(\tfrac{99}{100}\right) + 2(L(2) - L(3) + L(5)) \ , \end{aligned}$$

If efficiency is measured by the number of terms required for a given accuracy in (c) then x should be taken as close to 0 as possible; thus the problem of calculating a new value of L should be reduced to the calculation of $L(y)$ for y as close as possible to 1 . From this point of view the third formula given for $L(11)$ yields the most efficient calculation.

For $y = \frac{1 + x}{1 - x} = \frac{121}{120}$, $x = \frac{1}{221}$ and one term of the series already yields 4 decimal place accuracy for $L\left(\frac{121}{120}\right)$.

7. (a) Using upper and lower sums as estimates for $\int_1^n \frac{1}{t}\,dt$, obtain the inequality

$$\frac{1}{n} + L(n) < 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} < 1 + L(n) \ .$$

Take a subdivision of $[1,n]$ into $n - 1$ equal parts. Form the upper and lower Riemann sums. From $L < L(n) < U$, where

$$L = \sum_{k=1}^{n-1} \frac{1}{1+k} = \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

and

$$U = \sum_{k=1}^{n-1} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1},$$

the result follows.

(b) Estimate $\sum_{n=1}^{10^{100}} \frac{1}{n}$.

From Part (a),

$$10^{-100} + \log 10^{100} < \sum_{n=1}^{10^{100}} \frac{1}{n} < 1 + \log 10^{100};$$

hence,

$$\left| \sum_{n=1}^{10^{100}} \frac{1}{n} - 100 \log 10 \right| < 1,$$

(c) Prove that $\lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n}$ does not exist.

By Part (a),

$$\sum_{n=1}^{N} \frac{1}{n} > L(n) + \frac{1}{n} > L(n).$$

But, as we have remarked in the text, since $L(2^k) = kL(2)$, $L(n)$ cannot be bounded above, which excludes the possibility of a limit as $n \to \infty$.

8. Find $DL(|ax + b|)$ and obtain the corresponding integration formula over any interval $[h,k]$ where $ax + b \neq 0$ for any x in $[h,k]$

$$DL(|ax + b|) = \frac{D|ax + b|}{|ax + b|}$$

$$= \frac{D[(ax + b)\operatorname{sgn}(ax + b)]}{|ax + b|}$$

$$= \frac{a \operatorname{sgn}(ax + b)}{|ax + b|}$$

(since $D \operatorname{sgn}(ax + b) = 0$ wherever the derivative is defined)

$$= \frac{a}{ax + b} \text{ .}$$

The corresponding integration formula is

$$\int_h^k \frac{a}{ax + b}\, dx = L(|ax + b|)\Big|_h^k$$

$$= L\left|\frac{ak + b}{ah + b}\right|$$

where $ax + b \neq 0$ for any x in $[h,k]$.

Solutions Exercises 8-3

1. Verify Properties (3) - (5).

Property 3. If $u = L(a)$, $v = L(b)$, then

$$u + v = L(a) + L(b) = L(ab)$$

by Theorem 8-2. Consequently,

$$E(u + v) = EL(ab) = ab = E(u)E(v) \text{ .}$$

Property 4. If $u = L(a)$, $v = L(b)$, then

$$u - v = L(a) - L(b) = L(\tfrac{a}{b})$$

by Corollary 1 to Theorem 8-2. Consequently,

$$E(u - v) = EL(\tfrac{a}{b}) = \frac{a}{b} = \frac{E(u)}{E(v)} \text{ .}$$

Property 5. If $u = L(a)$ and r is any rational number, then by Corollary 3 to Theorem 8-2,

$$L(a^r) = r\,L(a)\ ;$$

hence,

$$EL(a^r) = E(ru)$$

and

$$a^r = E(ru)$$

or

$$\big(E(u)\big)^r = E(ru)\ .$$

2. Prove Equations (2) and (3) of Section 8-1 for real powers in general.

For the proof of Equation (2), use

$$\begin{aligned}
(a^x)^y &= \Big(E\big(x\ L(a)\big)\Big)^y && \text{(Definition 8-3)}\\
&= E\big(xy\ L(a)\big) && \text{(No. 1, Property 5)}\\
&= \big(EL(a)\big)^{xy} && \text{(No. 1, Property 5)}\\
&= a^{xy}\ .
\end{aligned}$$

For the proof of Equation (3), use

$$\begin{aligned}
(ab)^x &= E\big(xL(ab)\big) && \text{(Definition 8-3)}\\
&= E\big(xL(a) + xL(b)\big)\\
&= E\big(xL(a)\big)E\big(xL(b)\big) && \text{(Property 3)}\\
&= a^x b^x && \text{(Definition 8-3).}
\end{aligned}$$

3. Prove that the exponential function $x \longrightarrow a^x$, for positive a, $a \neq 1$, is strongly monotone.

Take $u > v$; use

$$a^u - a^v = E\big(uL(a)\big) - E\big(vL(a)\big)\ .$$

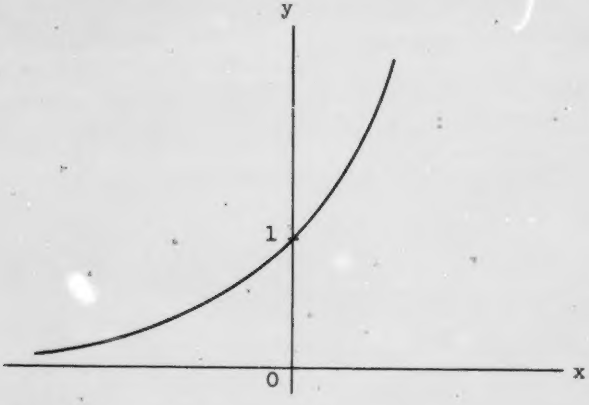
Since E and L are increasing functions, if $a > 1$, then $L(a) > 0$ and $a^u > a^v$ or $x \longrightarrow a^x$ is increasing, if $0 < a < 1$, then $L(a) < 0$ and $a^u < a^v$ or $x \longrightarrow a^x$ is decreasing.

4. Find the largest possible domain for each of the following functions:

(a)	$f : x \longrightarrow E(L(x))$.	$x > 0$
(b)	$f : x \longrightarrow L(E(x))$.	all x
(c)	$f : x \longrightarrow E(E(x))$.	all x
(d)	$f : x \longrightarrow L(L(x))$.	$x > 1$

5. Sketch the graph of the function given by $f(x)$.

(a)

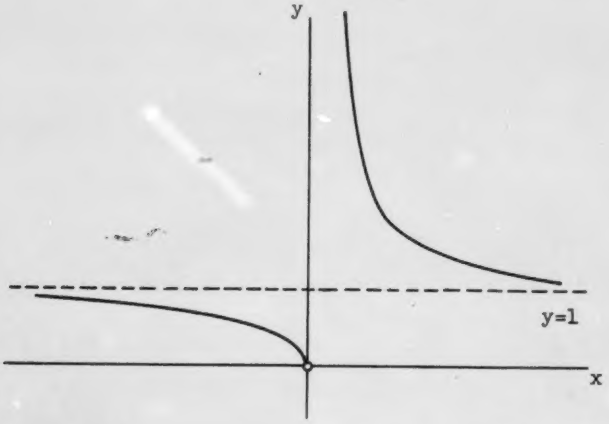


$f : x \longrightarrow 2^x$

(b) $f : x \longrightarrow 2^{-x}$. Reflect (a) in the y-axis.

(c) $f : x \longrightarrow 2^{x-1}$. Translate (a) one unit to the right.

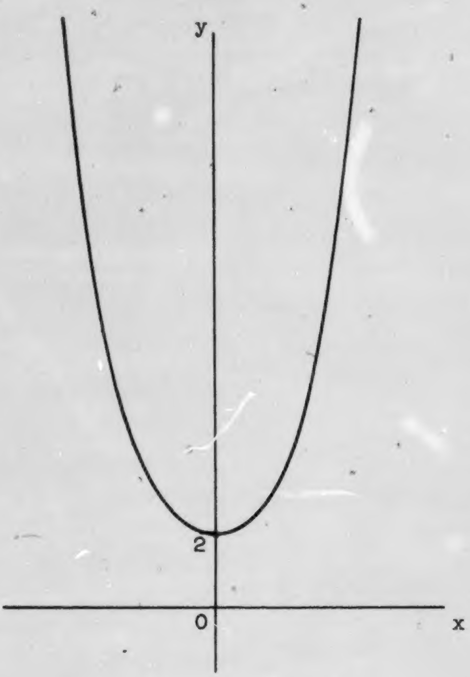
(d)



$f : x \longrightarrow 2^{1/x}$

(e) $f : x \longrightarrow 2^{-1/x}$. Reflect (d) in y-axis.

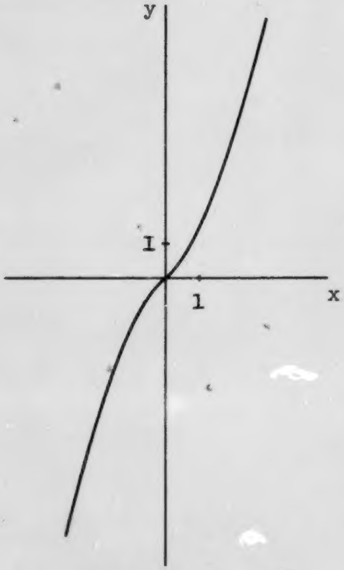
(f)



To obtain this graph add ordinates of the graphs of (a) and (b) .

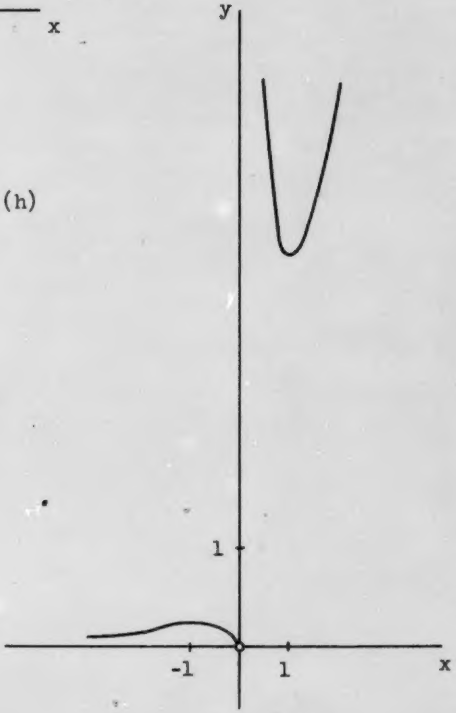
$f : x \longrightarrow 2^{x} + 2^{-x}$

(g)



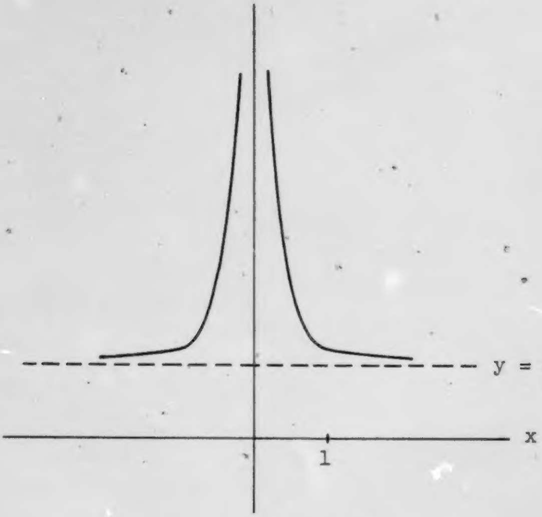
$f : x \longrightarrow 2^{x} - 2^{-x}$

(h)



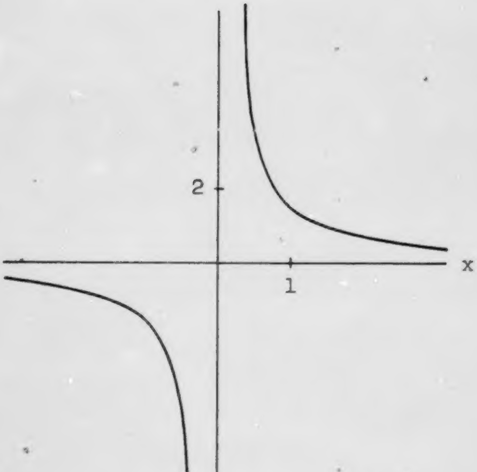
$f : x \longrightarrow 2^{x+\frac{1}{x}}$

(i)



$f : x \longrightarrow 2^{1/x} + 2^{-1/x}$

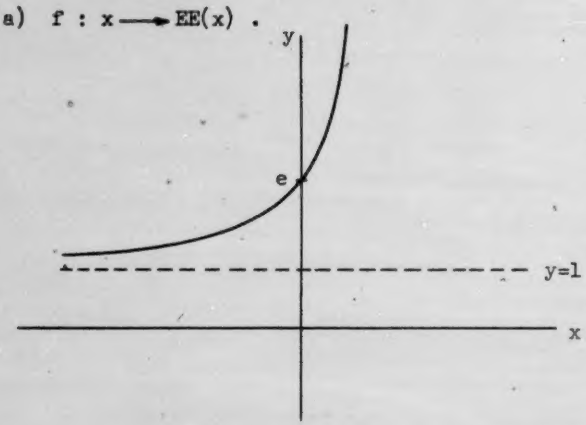
(j)



$f : x \longrightarrow 2^{1/x} - 2^{-1/x}$

6. Sketch the graph of

(a) $f : x \longrightarrow EE(x)$.



(b) $f : x \longrightarrow LL(x)$.

This is the inverse function to (a). Reflect in the line $y = x$.

7. In each of the following solve for x in terms of y .

(a) $y = L(\tan 2x)$; $x = \frac{1}{2}[\arctan E(y) + n\pi]$, n integer.

(b) $y = E(x^2 + 1)$; $x = \pm \sqrt{L(\frac{y}{e})}$, $y \geq e$.

(c) $y = E\big(x - L(y)\big)$; $x = 2L(y) = L(y^2)$, $y > 0$.

(d) $y = \log(x - \sqrt{x^2 - 1})$; $x = \dfrac{e^y + e^{-y}}{2}$.

(e) $y = E(x) - E(-x)$.

Set $e^x = u$. From $y = u - \frac{1}{u}$, $u^2 - uy - 1 = 0$; hence, since $u > 0$,

$$u = \frac{1}{2}(y + \sqrt{y^2 + 4})$$

and

$$x = \log[\frac{1}{2}(y + \sqrt{y^2 + 4})] .$$

8. Show that if $y = e^{\frac{1}{1-\log z}}$ and $z = e^{\frac{1}{1-\log x}}$, then $x = e^{\frac{1}{1-\log y}}$.

From $\log z = \frac{1}{1 - \log x}$,

$$y = \exp\left\{\frac{1}{1 - \frac{1}{1 - \log x}}\right\} = \exp\left\{\frac{\log x - 1}{\log x}\right\} ;$$

hence,

$$(\log x)(\log y) = \log x - 1 ,$$

and

$$\log x = \frac{1}{1 - \log y} ,$$

from which the result follows.

9. Prove that if f satisfies the functional equation $f(x + y) = f(x)f(y)$ for all x and y, and if $f(x) = 1 + x\, g(x)$ where $\lim_{x \sim 0} g(x) = 1$, then $f'(x)$ exists for every x and $f'(x) = f(x)$.

For any x,

$$\frac{f(x + h) - f(x)}{h} = \frac{f(x)f(h) - f(x)}{h}$$

$$= f(x) \cdot \frac{f(h) - 1}{h}$$

$$= f(x) \cdot g(h) .$$

Consequently,

$$f'(x) = \lim_{h \sim 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \sim 0} f(x) \cdot g(h)$$

$$= f(x) \cdot \lim_{h \sim 0} g(h)$$

$$= f(x) .$$

Solutions Exercises 8-4

1. Use logarithmic differentiation to evaluate each of the following.

(a) $D_x\left[\frac{(2x+3)^4}{(3x^2-1)^{1/2}}\right] = \frac{8(2x+3)^3}{(3x^2-1)^{1/2}} - \frac{3x(2x+3)^4}{(3x^2-1)^{3/2}}$.

(b) $D_x \sqrt[4]{\frac{x^4-1}{1+x^3}} = \sqrt[4]{\frac{x^4-1}{1+x^3}}\left(\frac{x^3}{x^4-1} - \frac{3x^2}{4(1+x^3)}\right)$.

(c) $D_x\left[\frac{\cos^3 x}{\sqrt[3]{1+2\sin x}}\right]^{1/2} = \left[\frac{\cos^3 x}{\sqrt[3]{1+2\sin x}}\right]^{1/2}\left(-\frac{3}{2}\tan x - \frac{\cos x}{3(1+2\sin x)}\right)$.

(d) $D_x\left[\sqrt{x^2-1}\;\sqrt[3]{1+6x^3}\right] = \left[\sqrt{x^2-1}\;\sqrt[3]{1+6x^3}\right]\left(\frac{x}{x^2-1} + \frac{6x^2}{1+6x^3}\right)$.

2. Find $D_x y$.

(a) $y = 3^x$; $\qquad 3^x \log 3$.

(b) $y = (\frac{1}{2})^x$; $\qquad -(\frac{1}{2})^x \log 2$.

(c) $y = e^{ax}$; $\qquad ae^{ax}$.

(d) $y = \log(1+x^2)$; $\qquad \frac{2x}{1+x^2}$.

3. Find $D_x y$.

(a) $y = (2^x)^2$; $\qquad 4^x \log 4$.

(b) $y = 2^{x^2}$; $\qquad 2x \cdot 2^{x^2} \cdot \log 2$.

(c) $y = e^{\cos x}$; $\qquad -e^{\cos x} \sin x$.

(d) $y = e^{\log\sqrt{x}}$; $\qquad \frac{1}{2\sqrt{x}}$.

(e) $y = \log\sqrt{e^x}$; $\qquad \frac{1}{2}$.

4. **Differentiate.**

(a) $y = x \log x - x$;	$\log x$.
(b) $y = e^x \sin x$;	$e^x(\sin x + \cos x)$.
(c) $y = \arctan e^x$;	$\dfrac{e^x}{1 + e^{2x}}$.
(d) $y = \log(\cos^2 x)$;	$-2 \tan x$.

5. **Differentiate.**

(a) $y = x^x$;	$\log y = x \log x$; $D_x y = x^x(1 + \log x)$.
(b) $y = x^{1/x}$;	$x^{1/x} \dfrac{(1 - \log x)}{x^2}$.
(c) $y = (\frac{1}{x})^x$;	$-\dfrac{(1 + \log x)}{x^x}$.
(d) $y = (\sin x)^x$;	$(\sin x)^x(x \cot x + \log \sin x)$.
(e) $y = x^{\cos x}$;	$x^{\cos x} (\frac{\cos x}{x} - \sin x \log x)$.
(f) $y = e^{x^x}$;	$e^{x^x} x^x(1 + \log x)$.
(g) $y = x^{x^x}$;	$x^{x^x} [x^{x-1} + x^x(\log x)(1 + \log x)]$.
(h) $y = x^{e^{-x}}$;	$x^{e^{-x}} e^{-x}(\frac{1}{x} - \log x)$.
(i) $y = \log \log x$;	$\dfrac{1}{(x \log x)}$.
(j) $y = \log \log \log x$;	$\dfrac{1}{(\log \log x)(x \log x)}$.

6. **Differentiate.** (First simplify, if possible.)

(a) $y = \log(\sec x + \tan x) = \log(1 + \sin x) - \log \cos x$;

$$y' = \frac{\cos x}{1 + \sin x} + \frac{\sin x}{\cos x} = \frac{1}{\cos x} = \sec x .$$

(b) $y = \frac{1}{2a} \log \frac{a - x}{a + x} = \frac{1}{2a}[\log(a - x) - \log(a + x)]$;

$$y' = \frac{1}{2a} [- \frac{1}{a - x} - \frac{1}{a + x}] = \frac{1}{x^2 - a^2} .$$

(c) $y = \log(x + \sqrt{a^2 + x^2}$;

$$y' = \frac{1 + \dfrac{x}{\sqrt{a^2 + x^2}}}{x + \sqrt{a^2 + x^2}} = \frac{1}{\sqrt{a^2 + x^2}} .$$

(d) $y = \sec x \tan x + \log(\sec x + \tan x)$;

$$y' = \sec^3 x + \sec x \tan^2 x + \sec x = 2 \sec^3 x .$$

(e) $y = \log \sqrt{x^3 + \sin^2(x^2 + e^{x^3})^3}$;

$$y' = \frac{3x^2 + 3(x^2 + e^{x^3})^2(2x + 3x^2e^{x^3})\sin 2(x^2 + e^{x^3})^3}{2[x^3 + \sin^2(x^2 + e^{x^3})^3]} .$$

7. Evaluate each of the following integrals.

(a) $\displaystyle\int_0^1 e^{2x}\,dx = \frac{1}{2}(e^2 - 1)$.

(b) $\displaystyle\int_1^3 \frac{2}{x}\,dx = \log 9$.

(c) $\displaystyle\int_{-1}^1 e^{ax}\,dx = \frac{1}{a}(e^a - e^{-a})$.

(d) $\displaystyle\int_0^1 2^x\,dx = \frac{1}{\log 2}$.

(e) $\displaystyle\int_0^a 2x\,e^{x^2}\,dx = e^{a^2} - 1$.

(f) $\displaystyle\int_0^a \frac{2x}{1 + x^2}\,dx = \log(1 + a^2)$.

(g) $\displaystyle\int_e^{e^e} \frac{dx}{x \log x} = \log \log x \Big|_e^{e^e} = 1$.

8. Show that if $e^x + e^y = e^{x+y}$ then $y' = -e^{y-x}$.

Differentiate implicitly:

$$e^x + e^y y' = e^x e^y y' + e^x e^y ;$$

hence,

$$e^y y'(e^x - 1) = e^x(1 - e^y)$$

and

$$y' = \frac{e^x - e^{x+y}}{e^{x+y} - e^y} = -\frac{e^y}{e^x} = -e^{y-x} .$$

9. Prove.

(a) $\lim\limits_{x\sim 0} \dfrac{\log(1 + x)}{x} = 1$

$$\begin{aligned}\lim_{x\sim 0} \frac{\log(1 + x)}{x} &= \lim_{x\sim 0} \frac{\log(1 + x) - \log 1}{x} \\ &= L'(1) = 1\end{aligned}$$

where $L = \log$.

(b) $\lim\limits_{x\sim 0} \dfrac{e^{cx} - 1}{x} = c$, c constant.

Take $f : x \longrightarrow e^{cx}$. Then

$$\lim_{x\sim 0} \frac{e^{cx} - 1}{x} = \lim_{x\sim 0} \frac{f(x) - f(0)}{x} = f'(0) = c .$$

(c) $\lim\limits_{x\sim 0} \dfrac{2 \log(1 + x) - 2x + x^2}{x^3} = \dfrac{2}{3}$

From Exercises 8-2, Number 6, for $|x| < 1$,

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + R_3$$

where $$|R_3| \le \frac{x^4}{5(1 - |x|)} .$$

Consequently,

$$\frac{2\log(1+x) - 2x + x^2}{x^3} = \frac{2}{3} + R_3$$

from which the result follows.

10. Assuming the existence of the limit in Equation (5), use the representation of $\log a$ as a limit to derive the addition property of logarithms (Theorem 8-2).

$$\log \frac{b}{a} = \lim_{h \to 0} \frac{1}{h}\left[\left(\frac{b}{a}\right)^h - 1\right]$$

But,

$$\frac{1}{h}\left[\left(\frac{b}{a}\right)^h - 1\right] = \frac{1}{h}\left[\frac{b^h - a^h}{a^h}\right]$$

$$= \frac{1}{a^h}\left[\frac{b^h - 1}{h} - \frac{a^h - 1}{h}\right] .$$

Consequently, from Equation (5) and the continuity of a^h,

$$\log \frac{b}{a} = \log b - \log a .$$

Now take $b = uv$, $a = v$ to obtain $\log u = \log uv - \log v$, from which the result follows.

11. Approximate the difference between a^x and a^{x+h} ($a > 0$ and $a \neq 1$) for large $|x|$ where $x < 0$.

By the Law of the Mean

$$a^{x+h} - a^x = h\, a^{\xi} \log a$$

where ξ lies between x and $x + h$. If $h > 0$ and $a > 1$, for example, then

$$a^x < a^{\xi} < a^{x+h}$$

and

$$h\, a^x \log a < a^{x+h} - a^x < h\, a^{x+h} \log a .$$

In this case, $\lim\limits_{x \to -\infty} a^{x+h} - a^x = 0$.

12. Prove that $e^x > 1 + x$ for all $x \neq 0$ and use this result to derive the following inequalities.

The function $f : x \longrightarrow e^x - x$ has the derivative $e^x - 1$ which is positive for $x > 0$ and negative for $x < 0$, hence has its minimum at $x = 0$. Consequently, $e^x - x > 1$.

Alternatively, the graph $y = e^x$ is everywhere flexed upward, hence lies above its tangent line $y = x + 1$ at $(0,1)$.

(a) $e^{-x} > 1 - x$, for all $x \neq 0$.

Replace x by $-x$ in the preceding result.

(b) $e^x > \frac{1}{1 - x}$, for $x > 1$.

Use (a), $\frac{1}{e^x} > 1 - x$, to obtain

$$1 > e^x(1 - x).$$

Since $1 - x$ is negative for $x > 1$ the sense of the inequality is reversed on division by $(1 - x)$.

(c) $e^x < \frac{1}{1 - x}$ for $x < 1$, $x \neq 0$.

Follows like (b) from (a) with $1 - x > 0$.

(d) $e^{-x} < \frac{1}{1 + x}$, for $x > -1$, $x \neq 0$.

Follows from $e^x > 1 + x$ for $x \neq 0$ with $1 + x > 0$ for $x > -1$.

13. Prove $\lim_{a \to -\infty} e^a = 0$.

From 12(c)

$$0 < e^a < \frac{1}{1 - a} \leq \frac{1}{1 + |a|} \quad \text{for } a < 0.$$

The result follows from the Squeeze Theorem.

14. Examine the behavior of f for small $|x|$ and very large $|x|$; use the information obtained to sketch the graph of f.

(a) $f : x \longrightarrow \dfrac{1}{1 - e^{1/x}}$

For small $|x|$, if $x > 0$, use $\lim\limits_{x \to 0^+} e^{-1/x} = 0$ to obtain $\lim\limits_{x \to 0^+} f(x) = 1$; if $x < 0$, use $\lim\limits_{x \to 0^-} e^{1/x} = 0$ to obtain $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \dfrac{e^{1/x}}{e^{1/x} - 1} = 0$. For large $|x|$, if $x > 0$ observe that $\lim\limits_{x \to \infty} e^{-1/x} = e^0 = 1$ and that the limit is approached monotomically from below, hence $f(x)$ increases without bound; if $x < 0$, the limit is approached from above, hence $-f(x)$ increases without bound as x approaches $-\infty$. For small $|u|$ observe further from the Law of the Mean, that

$$e^u - e^0 = e^{\xi} u, \qquad (0 < |\xi| < |u|).$$

Hence, for large $|x|$, $x = -\frac{1}{u}$

$$e^{-1/x} = 1 - \frac{e^{\xi}}{x},$$

Thus

$$f(x) = \frac{x}{e^{\xi}}$$

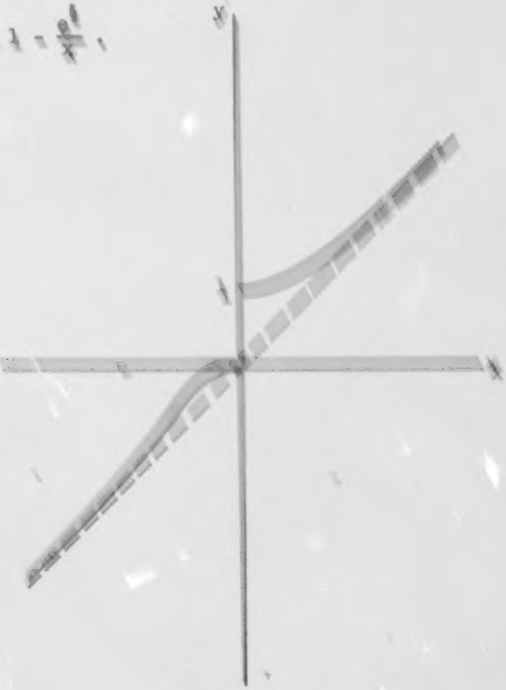
where

$$\lim_{x \to \pm\infty} e^{\xi} = 1,$$

It is easy to show that

$$\lim_{x \to \pm\infty} (f(x) - x) = 0$$

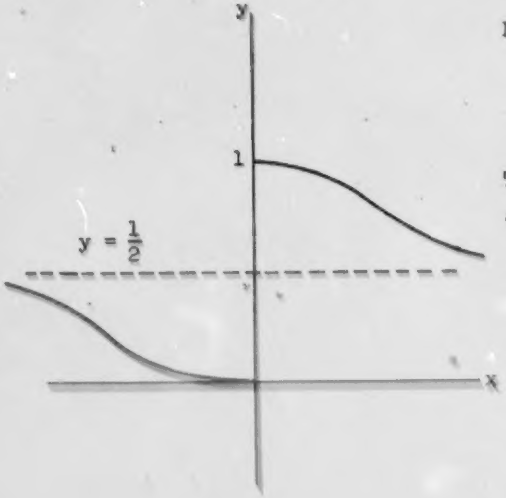
or that the line $y = x$ is an asymptote in both directions.



$f : x \longrightarrow \dfrac{1}{1 - e^{-1/x}}$

(b) $f : x \longmapsto \dfrac{e^{1/x}}{1 + e^{1/x}}$

Write
$$f(x) = \frac{1}{e^{-1/x} + 1}$$
from which it is easy to see that for small $|x|$
$$\lim_{x \to 0^+} f(x) = 1 \; , \; \lim_{x \to 0^-} f(x) = 0 \; .$$



For large $|x|$,
$$\lim_{x \to \pm\infty} f(x) = \frac{1}{2} \; .$$
Thus, the line $y = \frac{1}{2}$ is an asymptote in both directions.

15. The velocity v of a falling body at time t is given by
$$v = \frac{g}{k}(1 - e^{-kt})$$
where g and k are positive constants and the positive sense of motion is directed downward.

(a) Find the acceleration a at any time t. Show that $a = g - kv$.
$$a = \frac{dv}{dt} = ge^{-kt} \; ;$$
$$g - kv = g - \frac{kg}{k}(1 - e^{-kt}) = ge^{-kt}$$

(b) Find the distances that the body travels between $t = 0$ and $t = \tau$.
$$s = \int_0^\tau v \, dt = \frac{g}{k}\left[t + \frac{e^{-kt}}{k}\right]\Bigg|_0^\tau$$
$$= \frac{g}{k}[\tau + \frac{e^{-k\tau}}{k} - \frac{1}{k}]$$

(c) If g and t remain fixed determine

$$\lim_{k \to 0} v = \lim_{k \to 0} \frac{g}{k}(1 - e^{-kt}) .$$

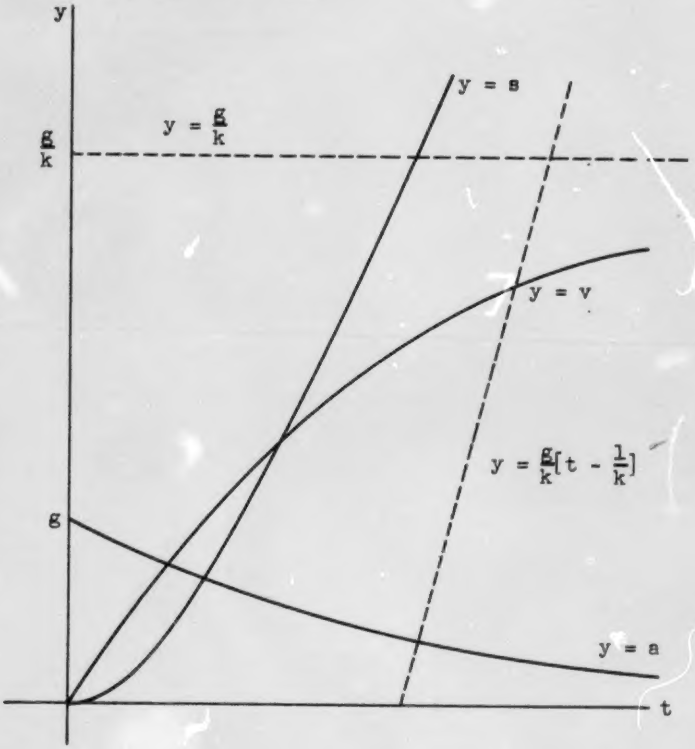
$$\frac{e^{kt} - 1}{k\,e^{kt}} = \frac{1 - e^{-kt}}{k} = \frac{f(k) - f(0)}{k}$$

where $f(k) = -e^{-kt}$. Consequently,

$$\lim_{k \to 0} v = \lim_{k \to 0} g \cdot \frac{f(k) - f(0)}{k} = g \cdot f'(0)$$

$$= gt .$$

(d) On the same set of axes, sketch the graphs of the functions $t \to s$, $t \to v$, and $t \to a$.



The graph $y = s$ has the asymptote $y = \frac{g}{k}[t - \frac{1}{k}]$; $y = v$, the asymptote $y = \frac{g}{k}$; $y = a$, the asymptote $y = 0$.

TC 8-5. The Differential Equations of e^x, $\sin x$, $\cos x$.

For many functions the relationship between the function and its derivatives, as expressed in a differential equation, yields significant information. In this section, we verify that the differential equations

(i) $$y' = cy, \qquad c \text{ constant},$$

(ii) $$(y')^2 + y^2 = k, \qquad k \text{ constant},$$

each subject to appropriate initial conditions, have unique solutions, the familiar exponential and circular functions, respectively. Through a study of the differential equations we seek to identify the properties that uniquely characterize these functions: in other words, we use the differential equations as a medium of investigation.

As students work through the exercises in this section they should reflect upon the ideas as previously encountered. The power of analysis is revealed when we establish results (for example, Exercises 8-5, Nos. 5, 6, 8) without appealing to geometrical intuition (as in Section 4-4).

The differential equation

$$y' = cy, \qquad c \text{ constant},$$

appears in many guises in Chapter 9 (for example, Equations (1) in Sections 9-3 (i), (ii), (iii) and 9-4).

In Sections 10-7, 10-8, and 10-9 we study the theory of differential equations more generally.

Solutions Exercises 8-5

1. For each of the following find the function f which satisfies the given differential equation subject to the given initial conditions.

(a) $y' = 2y$; $f(0) = 5$

By Theorem 8-5a, $y = 5e^{2x}$.

(b) $y'' + y = 0$; $f(0) = 2$, $f'(0) = -2$.

From Equation (10),

$$y = 2 \cos x - 2 \sin x .$$

2. Show that the function given by $y = f(x) = b \sin ax + d \cos ax$ satisfies the differential equation $y'' + a^2y = 0$ and the initial conditions $f(0) = d$, $f'(0) = ab$.

$$y'' = -a^2b \sin ax - a^2d \sin ax = -a^2y$$

$$f(0) = b \cdot \sin 0 + d \cdot \cos 0 = d$$

$$f'(0) = ab \cdot \cos 0 - ad \cdot \sin 0 = ab$$

3. Show that the function f given by

$$y = f(x) = e^x \sin x$$

satisfies the differential equation

$$y'' - 2y' + 2y = 0$$

and the initial conditions $f(0) = 0$, $f'(0) = 1$. Show also that the fourth derivative of $f(x)$ is proportional to $f(x)$.

$$y = e^x \sin x \qquad \text{(whence } f(0) = 0);$$

$$y' = e^x(\sin x + \cos x)$$

$$y'' = 2e^x \cos x ,$$

from which the result follows directly.

Differentiate in

$$y'' = 2y' - 2y$$

to obtain

$$y''' = 2y'' - 2y' = 2y' - 4y$$

and

$$y'''' = 2y'' - 4y' = -4y .$$

4. A particle moves along a line for 2 hours so that its velocity at any time t is given by $v = \frac{1}{2}e^{3t}$, in miles per hour. Find the displacement at the end of 2 hours and the distance traveled during the last hour.

The displacement at the end of two hours is

$$\int_0^2 \frac{1}{2} e^{3t} dt = \frac{e^{3t}}{6} \Big|_0^2 = \frac{e^6 - 1}{6} \text{ miles.}$$

The distance traveled during the last hour is

$$\int_1^2 \frac{1}{2} e^{3t} dt = \frac{e^{3t}}{6} \Big|_1^2 = \frac{e^6 - e^3}{6} \text{ miles.}$$

5. Derive Formula (11) for the inverse of ϕ directly from Equation (7).

We are given $u = \phi(x)$ where $\phi(0) = 0$, $\phi'(0) = 1$ and

$$(u')^2 + u^2 = C .$$

The conditions at $x = 0$ yield $C = 1$, hence

$$(u')^2 + u^2 = 1 ,$$

from which

$$u' = \pm\sqrt{1 - u^2} .$$

From the initial conditions, only the plus sign is possible; namely

$$u' = \sqrt{1 - u^2} .$$

If a solution of the original differential equation (4) exists in a neighborhood of $x = 0$ it is twice differentiable, hence has a continuous first derivative. The requirement that $u' = 1$ when $x = 0$ then guarantees that $u' > 0$ in some neighborhood of $x = 0$. It follows for $u = \phi(x)$ that ϕ is an increasing function and has an inverse g such that

$$u' = \phi'(x) = \frac{1}{g'(u)} ,$$

thus,

$$g'(u) = \frac{1}{\sqrt{1 - u^2}} .$$

Now, from the Fundamental Theorem

$$x = g(u) = A + \int_0^u \frac{dt}{\sqrt{1 - t^2}} ;$$

but $A = 0$ since $g(0) = 0$ and Formula (11) is obtained.

6. Prove the identity

$$u^2 + v^2 = 1$$

where $u = \phi(x)$ and $v = \psi(x)$ as defined above.

As observed in the solution of Number 5, ϕ satisfies (7) with $C = 1$:

$$u^2 + (u')^2 = 1 .$$

Insert $u' = \phi'(x) = \psi(x) = v$.

7. Prove that

$$u \leq \arcsin u \leq \frac{u}{1 - u^2}, \quad 0 \leq u < 1 .$$

Observe from the solution of Number 5 that

$$u' = \sqrt{1 - u^2} \leq 1 .$$

Integrate from 0 to x to obtain
(a) $u \leq x$,

From this inequality, provided $0 \leq x \leq 1$,

$$u' \geq \sqrt{1 - x^2} ;$$

hence

$$u' \geq \sqrt{(1 - x^2)^2} \geq 1 - x^2 ,$$

Integrate from 0 to x ;

$$u \geq x - \frac{x^3}{3}$$

hence, for x in $(0,1)$,

$$u \geq x\left(1 - \frac{x^2}{3}\right) \geq \frac{2x}{3} ,$$

or $x \leq \frac{3u}{2}$. Furthermore,

(b) $x \leq \dfrac{u}{1 - \frac{x^2}{3}} \leq \dfrac{u}{\left(1 - \frac{1}{3}\right)\left(\frac{3u}{2}\right)^2} \leq \dfrac{u}{1 - u^2}$;

Combine (a) and (b) to obtain

$$u \leq x \leq \frac{u}{1 - u^2}$$

which is the desired result.

8. Prove $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = \lim\limits_{x \to 0} \dfrac{\phi(x)}{x} = 1$.

$$\lim_{x \to 0} \frac{\phi(x)}{x} = \lim_{x \to 0} \frac{\phi(x) - \phi(0)}{x} = \phi'(0) = 1 .$$

9. Prove the result for differentiating the cosine; that is
$\psi'(x) = -\phi(x)$.

Since $\psi(x) = \phi'(x)$,

$$\psi'(x) = \phi''(x) = -\phi(x) .$$

10. Prove that ϕ and ψ as defined in this section are differentiable to all orders.

Since $\phi''(x) = \psi'(x) = -\phi(x)$ is differentiable, $\phi'''(x) = -\phi'(x)$ and $\phi'''(x)$ is differentiable. In the same way,

$$\phi^{iv}(x) = -\phi''(x) = \phi(x) .$$

In this way, the differential equation gives all higher order derivatives in terms of ϕ and ϕ' :

$$\phi^{(2n)} = (-1)^n\phi , \phi^{(2n+1)} = (-1)^n\phi' , (n = 0,1,2,\ldots) .$$

Furthermore, since $\phi' = \psi$

$$\psi^{(2n-1)} = (-1)^n\phi , \psi^{(2n)} = (-1)^n\phi' , (n = 1,2,3,\ldots) .$$

11. Prove that the sine is an odd function and that the cosine is even; i.e., that $\phi(-x) = -\phi(x)$ and $\psi(-x) = \psi(x)$.

Introduce the functions $\Phi : x \longmapsto \phi(-x)$ and $\Psi : x \longmapsto \psi(-x)$. Since

$$\Phi'(x) = -\phi'(-x) \text{ and } \Phi''(x) = \phi''(-x) = -\phi(-x) = -\Phi(x)$$

and the same relations hold for Ψ and ψ , it follows that Φ and Ψ are solutions of the differential equation (4) satisfying the initial conditions

$$\Phi(0) = 0 , \Phi'(0) = -1$$

and

$$\Psi(0) = 1 , \Psi'(0) = 0 .$$

It follows from Equation (10) that

$$\Phi(x) = -\phi(x) , \Psi(x) = \psi(x)$$

which is the result to be proved.

12. Prove the addition theorem for the sine:

$$\phi(a + b) = \phi(a)\psi(b) + \phi(b)\psi(a) .$$

(Hint: Use the fact that $\phi(a + x)$ is a solution of the differential equation (4).)

Observe for $f(x) = \phi(a + x)$ that

$$f''(x) = \phi''(a + x) = -\phi(a + x) = -f(x);$$

hence f is a solution of the differential equation (4). Furthermore,

$f(0) = \phi(a)$ and $f'(0) = \phi'(a) = \psi(a)$. From Equation (10), then,

$$f(x) = \phi(a + x) = \phi(a)\psi(x) + \psi(a)\phi(x) .$$

Take $x = b$ in this to obtain the stated result.

13. State and prove the corresponding addition theorem for the cosine.

$$\psi(a + b) = \psi(a)\psi(b) - \phi(a)\phi(b) .$$

Proceed as in Number 12.

14. Interpret the constant C in Equation (7) in terms of the amplitude of oscillation.

Any solution u of (4) can be written in the form

$$u = a\phi(x) + b\psi(x) ,$$

from which

$$u' = a\phi'(x) + b\psi'(x) = a\psi(x) - b\phi(x)$$

and

$$u^2 + (u')^2 = (a^2 + b^2)[\phi^2(x) + \psi^2(x)]$$
$$= a^2 + b^2 .$$

At a maximum or minimum of u , the derivative u' is zero and, therefore,

$$u^2 = a^2 + b^2 ;$$

hence,

$$u = \pm \sqrt{a^2 + b^2} = \pm A$$

where A is the "amplitude" of u . Consequently, $C = A^2$.

15. Discuss the existence and uniqueness of the solution of the following initial value problem

$$D^2f + \omega^2f = 0 ; f(0) = a , f'(0) = b .$$

First suppose $\omega \neq 0$. If f is a solution of this differential equation then, $g : x \longrightarrow f(\frac{x}{\omega})$ satisfies

$$g'(x) = \frac{1}{\omega} f'(\frac{x}{\omega}) \quad \text{and} \quad g'' = \frac{1}{\omega^2} f''(\frac{x}{\omega}) ,$$

and is therefore a solution of Equation (4):

$$D^2g + g = 0 .$$

Conversely, if g is a solution of Equation (4), then $f : x \longrightarrow g(\omega x)$ satisfies the present differential equation. Thus the given initial value problem is entirely equivalent to the problem for Equation (4) with the initial conditions

$$g(0) = a \ , \ g'(0) = \frac{b}{\omega} .$$

Thus the given initial value problem has the unique solution given by

$$\begin{aligned} f(x) &= a\psi(\omega x) + \frac{b}{\omega}\,\phi(\omega x) \\ &= a \cos \omega x + \frac{b}{\omega} \sin \omega x . \end{aligned}$$

If $\omega = 0$, the differential equation becomes

$$D^2f = Df' = 0 ,$$

hence f' is a constant function and by the initial condition on f' .

$$f'(x) = b .$$

It follows that $f(x) = bx + C$ where, from the initial condition on f , the constant is $C = a$. Thus

$$f(x) = a + bx .$$

This is the only possible solution and it clearly is a solution; therefore uniqueness and existence are established.

8-6. The Number e .

In this section we anticipate the development of Taylor's Theorem in Chapter 13. Here we obtain estimates for e^x which are used in future problem work. The discussion of $n!$ together with the development of upper and lower estimates for $n!$ provides the basis for an appreciation of the rapid growth of $n!$.

Solutions Exercises 8-6

1. (a) Using mathematical induction prove the result of Formula (3).

The result for $n = 1$, Formula (1), has been proved. If the result holds for $n = k$; that is, if

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$$1+x+\frac{x^2}{2}+\dots+\frac{x^k}{k!} \le e^x \le 1+x+\frac{x^2}{2}+\dots+\frac{x^{k-1}}{(k-1)!}+\frac{e^\alpha x^k}{k!}$$

then, from (2)

$$1+\left(x+\frac{x^2}{2}+\frac{x^3}{3\cdot 2}+\dots+\frac{x^{k+1}}{(k+1)(k!)}\right) \le e^x$$

$$\le 1+\left[x+\frac{x^2}{2}+\frac{x^3}{3\cdot 2}+\dots+\frac{x^k}{k[(k-1)!]}+\frac{e^\alpha x^{k+1}}{(k+1)(k!)}\right] .$$

which proves the result for $n = k + 1$.

(b) Obtain a result similar to Formula (3) for negative values of x and obtain an error estimate for your result like that of (5).

Suppose $-\alpha \le x \le 0$. In general, if $f(t) \le e^t \le g(t)$, then

$$\int_x^0 f(t)dt \le \int_x^0 e^t dt \le \int_x^0 g(t)dt ,$$

whence

$$1+\int_0^x g(t)dt \le e^x \le 1+\int_0^x f(t)dt .$$

Since $e^{-\alpha} \le e^t \le 1$ for t in $[-\alpha,0]$, it follows that

$$1+x \le e^x \le 1+e^{-\alpha}x \qquad (x \in [-\alpha,0]) .$$

Hence, by the same means

$$1+x+\frac{e^{-\alpha}x^2}{2} \le e^x \le 1+x+\frac{x^2}{2} .$$

Now prove in general, for all n that

$$\sum_{k=0}^{2n-1}\frac{x^k}{k!} \le e^x \le \sum_{k=0}^{2n-2}\frac{x^k}{k!}+\frac{e^{-\alpha}x^{2n-1}}{(2n-1)!}$$

and

$$\sum_{k=0}^{2n-1}\frac{x^k}{k!}+\frac{e^{-\alpha}x^{2n}}{(2n)!} \le e^x \le \sum_{k=0}^{2n}\frac{x^k}{k!} .$$

This result is established for $n = 1$. If it is true from $n = r$ then integrate the second inequality twice to obtain successively

$$1 + \sum_{k=0}^{2r} \frac{x^{k+1}}{(k+1)!} \le e^x \le 1 + \sum_{k=0}^{2r-1} \frac{x^{k+1}}{(k+1)!} + \frac{e^{-\alpha}x^{2r+1}}{(2r+1)!}$$

and

$$1 + x + \sum_{k=0}^{2r-1} \frac{x^{k+2}}{(k+2)!} + \frac{e^{-\alpha}x^{2r+2}}{(2r+2)!} \le e^x \le 1 + x + \sum_{k=0}^{2r} \frac{x^{k+2}}{(k+2)!} ;$$

but this is the stated result for $n = r + 1$.

For negative x, therefore,

$$\sum_{k=0}^{2p-1} \frac{x^k}{k!} \le e^x \le \sum_{k=0}^{2q} \frac{x^k}{k!}$$

for any natural numbers p, q. Consequently,

$$e^x = 1 + x + \frac{x^2}{2} + \dots \frac{x^n}{n!} + \epsilon_n(x)$$

where $\epsilon_n(x)$ is estimated for $x < 0$ as follows. If n is odd, $n = 2p - 1$, take $q = p$ above to obtain $0 \le \epsilon_n(x) \le \frac{x^{n+1}}{(n+1)!}$; if n is even, $n = 2q$, take $p = q + 1$ above to obtain $\frac{x^{n+1}}{(n+1)!} \le \epsilon_n(x) \le 0$. In either case,

$$|\epsilon_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} .$$

(c) Use the result of b to prove (7) for negative values of x.

It is sufficient to show that the error tolerance $\frac{|x|^{n+1}}{(n+1)!}$ can be made arbitrarily small by taking n large enough. Take $m > 2|x|$ and set $c = \frac{2^m |x|^m}{m!}$. Then, for $n > m$

$$\frac{|x|^{n+1}}{(n+1)!} = \frac{c}{2^m} \cdot \frac{|x|}{m+1} \cdot \frac{|x|}{m+2} \cdots \frac{|x|}{n+1}$$

$$< \frac{c}{2^{n+1}} .$$

2. Fill in the details to show that Formulas (4) and (5) follow from Formula (3).

Set $1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!} = e^x - \epsilon(x)$ in (3) to obtain

$$e^x - \epsilon(x) \leq e^x \leq e^x - \epsilon(x) + e^\alpha \frac{x^n}{n!} - \frac{x^n}{n!}$$

whence

$$0 \leq \epsilon(x) \leq \frac{(e^\alpha - 1)x^n}{n!} .$$

Now α is any number for which $x \leq \alpha$; in particular, α may be x itself; therefore e^α may be replaced by e^x in the preceding inequality. Finally, since $e < 3$ it follows from the increasing property of the power function $z \longrightarrow z^x$ for positive exponent that $e^x < 3^x$. Consequently,

$$0 \leq \epsilon(x) \leq \frac{(3^x - 1)x^n}{n!} .$$

3. Prove that the value for e given by (6) is correct to the indicated number of decimal places; i.e., show that the error in (6) is less than one half unit in the last place.

For the error $\epsilon(1)$ in cutting off the calculation at $n = 14$, we have

$$\epsilon(1) \leq \frac{2}{n!} \leq \frac{2}{14!} < \frac{2}{8.7 \times 10^{10}} < \frac{2}{8 \times 10^{10}} < 2.5 \times 10^{-11} .$$

The error of rounding off a number to eleven places is at most 5×10^{-12} and we have rounded 12 times to obtain a maximum rounding-off error of $12 \times 5 \times 10^{-12} = 6 \times 10^{-11}$. Thus the calculation yields a figure for e accurate within the sum of the two error estimates: 8.5×10^{-11} , better than 9 decimal place accuracy.

4. Use (9) and (10) to evaluate each of the following limits.

(a) $\lim\limits_{x \to \infty} (1 + \frac{2}{x})^x = \lim\limits_{\frac{x}{2} \to \infty} \left[(1 + \frac{2}{x})^{x/2}\right]^2 = e^2$

(b) $\lim\limits_{x \to \infty} (1 + \frac{1}{2x})^{x/3} = \lim\limits_{2x \to \infty} \left[(1 + \frac{1}{2x})^{2x}\right]^{1/6} = e^{1/6}$

(c) $$\lim_{h\sim 0}(1+2h)^{3/h} = \lim_{2h\sim 0}\left[(1+2h)^{1/2h}\right]^{6} = e^{6}$$

(d) $$\lim_{h\sim 0}(1+\sin h)^{2\csc h} = \lim_{\sin h\sim 0}\left[(1+\sin h)^{1/\sin h}\right]^{2} = e^{2}$$

(e) $$\lim_{h\sim\infty}\left(1-\frac{h}{2}\right)^{1/2h} = \lim_{-\frac{h}{2}\sim 0}\left[\left(1-\frac{h}{2}\right)^{-2/h}\right]^{-1/4} = e^{-1/4}$$

5. Evaluate $\lim\limits_{x\sim\infty} x(n^{1/x}-1)$.

This is equal to

$$\lim_{z\sim 0}\frac{n^{z}-1}{z} = D_{z}(n^{z})\Big|_{z=0}$$

$$= n^{z}\log n\Big|_{z=0} = \log n\ .$$

6. (a) A California savings and loan association offers an interest rate of 4.85% compounded continuously. What is the equivalent annual interest rate for money left on deposit one year?

If the number of interest periods were n we would have for the principal at the end of one year

$$P_{n} = P_{0}\left(1+\frac{.0485}{n}\right)^{n}$$

where P_0 is the initial deposit. The principal at the end of a year on the basis of continuous compounding is

$$P = \lim_{n\sim\infty}P_{n} = P_{0}\lim_{z\sim\infty}\left(1+\frac{1}{z}\right)^{.0485z} \qquad \left(z = \frac{n}{.0485}\right)$$

$$= P_{0}\left[\lim_{z\sim\infty}\left(1+\frac{1}{z}\right)^{z}\right]^{.0485}$$

$$= P_{0}\,e^{.0485}$$

$$= P_{0}\left(1 + .0485 + \frac{(.0485)^{2}}{2!} + \frac{(.0485)^{3}}{3!} + \dots\right)\ .$$

To three terms we have

$$P = P_0(1.0497) ,$$

equivalent to simple interest of 4.97% .

(b) How long does it take for an amount of money at the same interest rate (4.85% compounded continuously) to double itself?

In x years, the principle accumalted is $P_0e^{.0485x}$. We require $P = 2P_0$, hence $x = \frac{\log 2}{.0485} = \frac{0.693}{.0485} = 14.3$ years, approximately.

7. (a) Prove that $x \longrightarrow (1 + x)^{1/x}$ is a decreasing function.

Note that $\log[(1 + x)^{1/x}] = \frac{\log(1 + x)}{x}$ is the slope of the chord joining the points $(1,0)$ and $(1 + x, \log(1 + x))$ on the graph $y = \log x$. Since the graph is flexed downward, the result follows from Exercises 5-5, Number 11.

Alternatively, differentiate to obtain

$$D_x[(1 + x)^{1/x}] = [x - (1 + x)\log(1 + x)]\frac{(1 + x)^{1/x - 1}}{x^2}$$

$$= [x - (1 + x)\log(1 + x)]p(x)$$

where $p(x) > 0$ and the sign of the derivative is therefore determined by the factor in brackets. From

$$D_x[x - (1 + x)\log(1 + x)] = -\log(1 + x)$$

the bracketed factor is seen to have the maximum value 0 at $x = 0$. It follows that the original function has a negative derivative everywhere except at 0 and, hence, that the function is decreasing on its domain (which does not contain the point 0).

(b) Which is greater

$$1000^{1001} \quad \text{or} \quad 1001^{1000} \text{ ?}$$

From Part (a),

$$e = \lim_{z \sim 0} (1 + z)^{1/z} > (1 + x)^{1/x}, \qquad \text{for } x > 0 .$$

Consequently, for $x = \frac{1}{1000}$,

$$1000 > e > \left(\frac{1001}{1000}\right)^{1000},$$

from which

$$1000^{1001} > 1001^{1000} .$$

(c) Which is greater,

$$1{,}000{,}000^{1{,}000{,}001} \quad \text{or} \quad 1{,}000{,}010^{1{,}000{,}000} ?$$

From Part (a),

$$\left(\frac{1{,}000{,}010}{1{,}000{,}000}\right)^{1{,}000{,}000} = \left[\left(1 + \frac{1}{100{,}000}\right)^{100{,}000}\right]^{10}$$

$$< e^{10} \le (e^2)^5 < 9^5 < 10^6$$

from which

$$1{,}000{,}000^{1{,}000{,}001} > 1{,}000{,}010^{1{,}000{,}000} .$$

8. Show that for $n > 9$

$$\sqrt{n}^{\sqrt{n+1}} > \sqrt{n+1}^{\sqrt{n}} .$$

Take square roots in the inequality

$$\left(1 + \frac{1}{n}\right)^n < e$$

to obtain

$$\left(\sqrt{\frac{n+1}{n}}\right)^n$$

then raise to the power $\frac{1}{\sqrt{n}}$ to obtain

$$\sqrt{\frac{n+1}{n}}^{\sqrt{n}} < \sqrt{e}^{1/\sqrt{n}}$$

and hence

$$\left(\sqrt{n+1}\right)^{\sqrt{n}} < \sqrt{e}^{1/\sqrt{n}} \sqrt{n}^{\sqrt{n}} .$$

To complete the proof it is necessary to show that

$$\sqrt{e}^{1/\sqrt{n}} \; \sqrt{n}^{\sqrt{n}} < \sqrt{n}^{\sqrt{n+1}} .$$

This last inequality holds if and only if

$$\sqrt{e}^{1/\sqrt{n}} < \sqrt{n}^{\sqrt{n+1} - \sqrt{n}}$$

or

$$e < n^{(\sqrt{n+1} - \sqrt{n})\sqrt{n}} .$$

Now

$$(\sqrt{n + 1} - \sqrt{n}) = \frac{1}{\sqrt{n + 1} + \sqrt{n}} > \frac{1}{2\sqrt{n + 1}} .$$

We need only choose n so large that

$$\text{(1)} \qquad e \leq n^{\frac{1}{2}\sqrt{\frac{n}{n+1}}} < n^{(\sqrt{n+1} - \sqrt{n})\sqrt{n}}$$

to obtain the desired result. Since $n^{\frac{1}{2}\sqrt{\frac{n}{n+1}}} \approx n^{1/2}$ for large n, very likely $n = 9$ or some slightly larger integer will suffice. Since $\frac{n}{n + 1} = 1 - \frac{1}{n + 1}$ is an increasing function of n, once we have obtained an integer n for which the inequality (1) holds, we know it holds for all larger integers. Taking logarithms we see that (1) is equivalent to

$$\frac{1}{2} \sqrt{\frac{n}{n + 1}} \log n \geq 1 .$$

From Exercises 8-2, Number 6(c), we have for $n = 9$

$$\begin{aligned} \frac{1}{2} \sqrt{\frac{n}{n + 1}} \log n &= \frac{1}{2} \sqrt{\frac{9}{10}} \log 3^2 \\ &= \frac{3}{10} \sqrt{10} \log 3 \\ &> \frac{3}{10} \times 3.16 \times 1.09 > 1.03 . \end{aligned}$$

We conclude that the result to be proved holds for $n \geq 9$.

9. **Show how to obtain an approximation of**

$$\int_0^t e^{x^2}\,dx\,, \qquad 0 < t < 1\,.$$

Use Equation (4) to obtain an estimate of the error of approximation.

Since $0 \le x^2 < 1$ we have from Equations (4) and (5)

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \dots + \frac{x^{2n}}{n!} + \epsilon(x^2)$$

where $0 \le \epsilon(x^2) < \frac{2}{n!}$. Integrate to obtain

$$\int_0^t e^{x^2}\,dx = t + \frac{t^3}{3} + \frac{t^5}{2!5} + \dots + \frac{t^{2n+1}}{n!(2n+1)} + R$$

where the remainder R satisfies

$$0 \le R < \frac{2t}{n!} < \frac{2}{n!}\,.$$

10. **Prove that $n! > 3(\frac{n}{e})^n$ for $n > 1$. (Hint: Use mathematical induction.)**

For $n = 2$,

$$3(\tfrac{n}{e})^n = 3(\tfrac{2}{e})^2 < 3(\tfrac{2}{2.5})^2 \le 3(\tfrac{4}{5})^2$$

$$\le \frac{48}{25} < 2!$$

If, for $n = k$,

$$k! > 3(\tfrac{k}{e})^k$$

then

$$\begin{aligned}(k+1)! &> 3(k+1)(\tfrac{k}{e})^k \\ &\ge 3e(\tfrac{k+1}{e})^{k+1}(\tfrac{k}{k+1})^k \\ &\ge \frac{3e}{(1+\frac{1}{k})^k}(\tfrac{k+1}{e})^{k+1} \\ &\ge \frac{3e}{e}(\tfrac{k+1}{e})^{k+1} \qquad \text{(from No. 7(a))} \\ &\ge 3(\tfrac{k+1}{e})^{k+1}\,.\end{aligned}$$

TC8-7. The Hyperbolic Functions.

The hyperbolic functions and their inverses are mentioned in Section 10-1 and Exercises 10-2, Number 3. In Section 10-3 we use hyperbolic substitution to transform an integral into an integral of a rational combination of hyperbolic functions. They appear again (Exercises 10-8) in the solution of second order linear differential equations with constant coefficients.

Solutions Exercises 8-7

1. Derive Formulas (1)-(6) from the definitions of the hyperbolic functions.

(1) $$\cosh^2 x - \sinh^2 x = \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4}$$
$$= 1$$

(2) $$\sinh x \cosh y + \cosh x \sinh y = \frac{1}{4}[e^{x+y} + e^{x-y} - e^{-x+y} - e^{-(x+y)}]$$
$$+ \frac{1}{4}[e^{x+y} - e^{x-y} + e^{-x+y} - e^{-(x+y)}]$$
$$= \frac{e^{x+y} - e^{-(x+y)}}{2} = \sinh(x + y) .$$

(3) $$\cosh x \cosh y + \sinh x \sinh y = \frac{1}{4}[e^{x+y} + e^{x-y} + e^{y-x} + e^{-(x+y)}]$$
$$+ \frac{1}{4}[e^{x+y} - e^{x-y} - e^{y-x} + e^{-(x+y)}]$$
$$= \frac{e^{x+y} + e^{-(x+y)}}{2} = \cosh(x + y) .$$

(4) $$D \sinh x = D\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x .$$

(5) $$D \cosh x = D\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x .$$

(6) $$D \tanh x = D\frac{\sinh x}{\cosh x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} .$$
$$= 1 - \tanh^2 x .$$

2. Apply the methods of Section 5-8 to discuss the graphs of the hyperbolic functions.

For $\sinh x$:

$$D \sinh x = \cosh x > 0$$

so that sinh is an increasing function. Therefore $\sinh x$ has only one zero, given by $x = 0$. $D^2 \sinh x = \sinh x$; therefore $(0,0)$ is a point of inflection, the graph $y = \sinh x$ being flexed upward for $x > 0$, downward for $x < 0$.

For $\cosh x$:

$$D \cosh x = \sinh x \ , \ D^2 \cosh x = \cosh x \ .$$

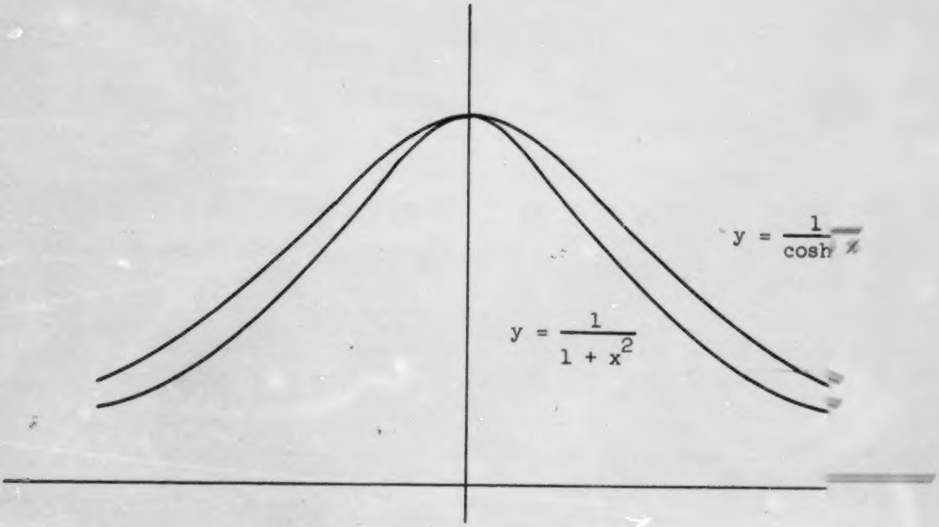
Therefore $\cosh x$ has a minimum at $x = 0$ and the graph is flexed upward.

For $\tanh x$:

$$D \tanh x = 1 - \tanh^2 x \ ;$$

hence the derivative is everywhere positive and tanh is an increasing function. Thus the zero at $x = 0$ is unique. $D^2 \tanh x = -2 \tanh x (1 - \tanh^2 x)$. There is a point of inflection at $(0,0)$ separating the interval of upward flexure $(x < 0)$ from the interval of downward flexure $(x > 0)$.

3. Sketch the graphs of $y = \frac{1}{\cosh x}$ (that is, $y = \operatorname{sech} x$) and $y = \frac{1}{1 + x^2}$ on the same set of axis.



4. Find $\lim\limits_{x \to 0} \dfrac{\sinh x}{x}$.

$$\begin{aligned} \lim_{x \to 0} \frac{\sinh x}{x} &= \lim_{x \to 0} \frac{\sinh x - \sinh 0}{x} \\ &= D_x \sinh x \Big|_{x=0} \\ &= \cosh 0 \\ &= 1 \end{aligned}$$

5. Differentiate.

(a) $\dfrac{1}{\tanh^2 x}$

$$D\left(\frac{1}{\tanh^2 x}\right) = \frac{-2(1 - \tanh^2 x)}{\tanh^3 x} = 2 \coth x - 2 \coth^3 x \ .$$

(b) $\sinh (2 \log x)$

From $\sinh (2 \log x) = \frac{1}{2}(x^2 - x^{-2})$,

$$D \sinh (2 \log x) = x + \frac{1}{x^3} \ .$$

(c) $e^x \sinh x$

From $e^x \sinh x = \frac{1}{2}(e^{2x} - 1)$,

$$D(e^x \sinh x) = e^{2x} \ .$$

(d) $\dfrac{\cosh x + \sinh x}{\cosh x - \sinh x}$

$$\begin{aligned} D\left(\frac{\cosh x + \sinh x}{\cosh x - \sinh x}\right) &= D[(\cosh x + \sinh x)^2] \\ &= De^{2x} = 2e^{2x} \end{aligned}$$

6. Show that $\sinh y + \cosh y = e^y$, for each y, and verify that $(\sinh x + \cosh x)^n = \sinh nx + \cosh nx$.

$$\sinh y + \cosh y = \frac{e^y - e^{-y}}{2} + \frac{e^y + e^{-y}}{2}$$
$$= e^y ;$$

hence,

$$(\sinh x + \cosh x)^n = (e^x)^n = e^{nx} = \sinh nx + \cosh nx .$$

7. Show that $\arctan \sinh x = \arcsin \tanh x$.

$$D \arctan \sinh x = \frac{\cosh x}{1 + \sinh^2 x} = \frac{1}{\cosh x}$$

$$D \arcsin \tanh x = \frac{1 - \tanh^2 x}{\sqrt{1 - \tanh^2 x}} = \sqrt{1 - \tanh^2 x} = \frac{1}{\cosh x}$$

By Corollary C to Theorem 5-4a,

$$\arctan \sinh x = \arcsin \tanh x + C$$

where C is constant. Take $x = 0$ to obtain $C = 0$.

8. Show that $D_x(2 \arctan e^x) = D_x \arctan \sinh x$.

Does $2 \arctan e^x = \arctan (\sinh x)$? Justify your answer.

$$Dx(2 \arctan e^x) = \frac{2e^x}{1 + e^{2x}} = \frac{2}{e^{-x} + e^x} = \frac{1}{\cosh x}$$

$$D_x \arctan \sinh x = \frac{1}{\cosh x}, \qquad \text{from Number 7.}$$

No, $2 \arctan e^x = \arctan \sinh x + C$ where C is constant; take $x = 0$ to obtain $C = \frac{\pi}{2}$.

9. Calculate the inverse functions of sinh, cosh, and tanh in terms of logarithms.

The inverse functions are indicated by the prefix "arg"; for example, argsinh or "argument of the hyperbolic sine."

Set $y = \sinh x = \frac{e^x - e^{-x}}{2}$; then

$$e^{2x} - 2y\,e^x - 1 = 0 ,$$

whence,

$$e^x = y + \sqrt{y^2 + 1}$$

where the sign of the radical is determined by the sign of e^x. Consequently,

$$x = \operatorname{argsinh} y = \log (y + \sqrt{y^2 + 1}) .$$

Similarly, from $y = \cosh x = \frac{e^x + e^{-x}}{2}$

$$x = \operatorname{argcosh} y = \log(y + \sqrt{y^2 - 1}) .$$

Here the choice of sign is a convention; cosh is not monotone and either sign is possible. The inverse is chosen positive. The domain of argcosh is $\{y : y \geq 1\}$.

From $y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$,

$$e^{2x} = \frac{1 + y}{1 - y} ;$$

whence

$$x = \operatorname{argtanh} y = \frac{1}{2} \log \frac{1 + y}{1 - y} .$$

The domain of argtanh is $\{y : |y| < 1\}$.

10. Obtain the derivatives of the inverse hyperbolic functions using (1) and note how they differ from the derivatives of the corresponding inverse trigonometric functions. Observe that the derivatives of the inverse hyperbolic functions are algebraic functions.

Apply Theorem 4-3 on the differentiation of inverse functions:

$$\begin{aligned} D \operatorname{argsinh} x &= \frac{1}{\cosh(\operatorname{argsinh} x)} \\ &= \frac{1}{\cosh[(\operatorname{sgn} x)\operatorname{argcosh} \sqrt{1 + x^2}} \\ &= \frac{1}{\cosh(\operatorname{argcosh} \sqrt{1 + x^2})} \\ &= \frac{1}{\sqrt{1 + x^2}} , \end{aligned}$$

$$\begin{aligned} D \operatorname{argcosh} x &= \frac{1}{\sinh(\operatorname{argcosh} x)} \qquad (x > 1) \\ &= \frac{1}{\sinh(\operatorname{argsinh} \sqrt{x^2 - 1})} \\ &= \frac{1}{\sqrt{x^2 - 1}} \, . \end{aligned}$$

$$\begin{aligned} D \operatorname{argtanh} x &= \frac{1}{1 - \tanh^2(\operatorname{argtanh} x)} \\ &= \frac{1}{1 - x^2} \qquad (|x| < 1) \, . \end{aligned}$$

11. Find the length of the catenary $y = a \cosh \frac{x}{a}$ between $x = 0$ and $x = b$.

The arclength is

$$\begin{aligned} \int_0^b \sqrt{1 + (y')^2}\, dx &= \int_0^b \sqrt{1 + \sinh^2 \frac{x}{a}}\, dx \\ &= \int_0^b \cosh \frac{x}{a}\, dx = a \sinh \frac{x}{a} \Big|_0^b \\ &= a \sinh \frac{b}{a} \, . \end{aligned}$$

12. Use (3) of Section 8-6 to find upper and lower polynomial bounds for $\sinh x$ and $\cosh x$.

From Section 8-6, Equation (3), for $0 \le x \le \alpha$,

$$p(x) \le e^x \le p(x) + \frac{e^{\alpha} x^n}{n!}$$

where

$$p(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} \, .$$

For negative argument, use the result of Exercises 8-6, Number 1(b):

$$q(x) \le e^{-x} \le q(x) + \frac{x^{2m+2}}{(2m + 2)!} \, , \qquad (0 \le x) \, ,$$

where

$$q(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots - \frac{x^{2m+1}}{(2m + 1)!} \, .$$

Take $n = 2m$, add, and divide by 2 , to obtain

$$\left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2m}}{(2m)!}\right] - \frac{1}{2}\,\frac{x^{2m+1}}{(2m+1)!} \le \cosh x$$

$$\le \left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2m}}{(2m)!}\right] + \frac{e^{\alpha}x^{2m}}{(2m)!} - \frac{1}{2}\,\frac{x^{2m+1}}{(2m+1)!} + \frac{1}{2}\,\frac{x^{2m+2}}{(2m+2)!}\,.$$

Subtract and divide by 2 to obtain for sinh

$$\left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2m-1}}{(2m-1)!}\right] + \frac{1}{2}\,\frac{x^{2m+1}}{(2m+1)!} \le \sinh x$$

$$\le \left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2m-1}}{(2m-1)!}\right] + \frac{e^{\alpha}x^{2m}}{(2m)!} + \frac{1}{2}\,\frac{x^{2m+1}}{(2m+1)!} - \frac{1}{2}\,\frac{x^{2m+2}}{(2m+2)!}$$

Use the even and odd symmetry of the two functions to obtain the bounds for negative x .

13. (a) Obtain a formula for $\tanh(a + b)$ in terms of hyperbolic tangents.

From Formulas (2) and (3),

$$\tanh(a+b) = \frac{\sinh(a+b)}{\cosh(a+b)} = \frac{\sinh a \cosh b + \cosh a \sinh b}{\cosh a \cosh b + \sinh a \sinh b}$$

$$= \frac{\dfrac{\sinh a}{\cosh a} + \dfrac{\sinh b}{\cosh b}}{1 + \dfrac{\sinh a}{\cosh a}\,\dfrac{\sinh b}{\cosh b}} = \frac{\tanh a + \tanh b}{1 + \tanh a \tanh b}\,.$$

(b) Give $\tanh 4x$ in terms of $\tanh x$.

Apply the preceding result as follows;

$$\tanh 2x = \tanh(x + x) = \frac{2 \tanh x}{1 + \tanh^2 x}\,;$$

$$\tanh 4x = \frac{2 \tanh 2x}{1 + \tanh^2 2x}$$

$$= \frac{4 \tanh x}{1 + \tanh^2 x} \Big/ \left[1 + \frac{4 \tanh^2 x}{(1 + \tanh^2 x)^2}\right]$$

$$= \frac{4 \tanh x\,(1 + \tanh^2 x)}{\tanh^4 x + 6 \tanh^2 x + 1}\,.$$

14. The differential equation

(a) $D^2f - f = 0$ is satisfied by cosh and sinh .
Prove the uniqueness of the solution of (a) under the initial condition
(b) $f(0) = a$, $f'(0) = b$
and express the solution in terms of sinh and cosh.
(Hint: Show that if f is a solution of (a), then $g = Df - f$ is a solution of $Dg + g = 0$.)

If f is a solution of (a) then $g = Df - f$ satisfies

$$\begin{aligned} Dg + g &= (D^2f - Df) + (Df - f) \\ &= D^2f - f = 0 \end{aligned}$$

together with the initial condition

$$g(0) = f'(0) - f(0) = b - a .$$

Thus, from Theorem 8-5(a), g is the unique solution of the initial value problem

$$\text{(c)} \qquad \begin{cases} Dg + g = 0 , \\ g(0) = f'(0) - f(0) = b - a , \end{cases}$$

Consequently, a solution f of the original problem satisfies the initial value problem

$$\text{(d)} \qquad \begin{cases} Df - f = g \\ f(0) = a \end{cases}$$

where g is fixed by (c). If h is any solution of (d) , then for $\phi = h - f$ we have

$$\begin{cases} D\phi - \phi = 0 , \\ \phi(0) = 0 , \end{cases}$$

but from Theorem 8-5, again, the solution of this initial value problem is unique. It follows that $\phi = 0$ is the only possible solution; hence $h = f$.

Clearly $f(x) = a \cosh x + b \sinh x$ is a solution of the original initial value problem, and by the uniqueness proof it is the only one.

Solutions Miscellaneous Exercises

1. Evaluate:

(a) $$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{1+\frac{k}{n}}\right)\frac{1}{n}\,.$$

[Hint: Use Theorem 6-3c.]

This is the limit of a Riemann sum for the function $\frac{1}{x}$ over the interval $[1,2]$ with a subdivision of the interval into n equal parts.

Answer: $\int_1^2 \frac{1}{x}\,dx = \log 2$.

(b) $$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{n+a+bk}\,,\quad a\,,\ b>0\,.$$

The sum can be approximated by a Riemann sum as follows:

$$\sum_{k=0}^{n}\frac{1}{n+a+bk}=\frac{1}{b}\sum_{k=0}^{n}\left(\frac{1}{1+\frac{bk}{n}}\right)\frac{h}{n}+a_n$$

where

$$|\epsilon_n| = \left|\sum_{k=0}^{n}\frac{1}{n+a+bk}-\frac{1}{n+bk}\right|$$

$$=\left|\sum_{k=0}^{n}\frac{-a}{(n+bk)(n+a+bk)}\right|$$

$$=\frac{1}{n}\sum_{k=0}^{n}\frac{a}{(1+\frac{bk}{n})(1+\frac{a}{n}+\frac{bk}{n})}\frac{1}{n}$$

$$\leq\frac{1}{nb}\left\{\sum_{k=1}^{n}\frac{a}{(1+\frac{bk}{n})^2}\frac{b}{n}\right\}+\frac{a}{n^2}\,;$$

here, at the last step, the term $\frac{a}{n}$ was dropped from the second factor in the denominator and the first term of the sum separated out. The expression in braces is the Riemann lower sum for $f : x \to \frac{a}{x^2}$ over the interval $[1,1+b]$ with the interval subdivided into n equal parts. Consequently,

$$|\epsilon_n| \le \frac{a}{n(b+1)} + \frac{a}{n^2}$$

and $\lim\limits_{n\to\infty} \epsilon_n = 0$. It follows that

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{n+a+bk} = \lim_{n\to\infty} \left\{ \left(\frac{1}{b} \sum_{k=1}^{n} \frac{1}{1+\frac{bk}{n}} \cdot \frac{b}{n} \right) + \frac{1}{n} + \epsilon_n \right\}$$

$$= \frac{1}{b} \int_1^{1+b} \frac{1}{x}\, dx = \frac{1}{b} \log(1+b) .$$

2. Find the mean value (Exercises 6-4, No. 20) of the function

$$f : x \longrightarrow \frac{1}{x}$$

on the interval $[1,2]$.

The mean value $f(u)$ satisfies

$$f(u) = \frac{1}{2-1} \int_1^2 \frac{1}{x}\, dx ;$$

hence,

$$f(u) = \frac{1}{u} = \log 2 .$$

3. If $F(x) = \int_0^{\sqrt{x}} e^{-t^2}\, dt$, $x > 0$, find $F'(x)$.

$$F'(x) = e^{-(\sqrt{x})^2} D\sqrt{x} = \frac{e^{-x}}{2\sqrt{x}} .$$

4. The region bounded by the curve $y = e^{-x}$ and the lines $y = 0$, $x = 0$, $x = 10$ is rotated about the x-axis. Compute to three significant digits the volume of the solid of revolution so generated. Justify any approximations you use.

$$V = \pi \int_0^{10} y^2 dx = \pi \int_0^{10} e^{-2x}\,dx$$

$$= -\frac{\pi}{2} e^{-2x} \Big|_0^{10}$$

$$= \frac{\pi}{2}(1 - e^{-20})$$

$$\approx \frac{3.1416}{2} \approx 1.5708 \ .$$

The term e^{-20} is negligible to the required accuracy. To three significant digits,

$$V \approx 1.571 \ .$$

5. Find all solutions of the equation $e^x = e$. Justify your answer.

The only solution is $x = 1$ since exp is an increasing function.

6. Find an equation of the tangent line to the graph of $y = e^x$

(a) at $(0,1)$. $\qquad y = x + 1$

(b) at (a,e^a). $\qquad y = e^a(x - a + 1)$

(c) that passes through the origin.

From (b), choose a so that $0 = e^a(0 - a + 1)$, i.e., so that $a = 1$. The point of tangency is $(1,e)$ and the equation of the tangent line is $y = ex$.

7. Find an equation of the only tangent line to the graph of $y = \log x$ that passes through the origin. Show that there is only one such line.

Note that the graph $y = \log x$ is obtained from the graph $y = e^x$ by reflection in the line. In such a reflection, the origin is fixed; hence, by Number 6(c), the tangent line is $y = \frac{x}{e}$.

8. Find an equation of the tangent line to the curve $y = \log(\tan x)$ at the point of inflection (x_0,y_0) where $0 < x_0 < \frac{\pi}{2}$.

$$y' = \frac{1}{\tan x \cos^2 x} = \frac{1}{\sin x \cos x} = \frac{2}{\sin 2x} .$$

$$y'' = -\frac{4 \cos 2x}{\sin^2 2x} .$$

At $x = \frac{\pi}{4}$, y'' changes sign and there is a point of inflection. The tangent line is $y = 2(x - \frac{\pi}{4})$.

9. Find equations of the tangent line and normal line, respectively, to the graph of f at the given point.

(a) $f : x \longrightarrow \frac{1}{1 + e^{-x}}$, at $x = 0$;

tangent, $y = \frac{x}{4} + \frac{1}{2}$; normal, $y = -4x + \frac{1}{2}$.

(b) $f : x \longrightarrow x^x$, at $x = \frac{1}{e}$; tangent, $y = e^{-1/e}$; normal, $x = \frac{1}{e}$.

10. At any point where two curves intersect, the angle between their respective tangent lines is called their "angle of intersection." For each point of intersection of the graphs of $f : x \longrightarrow e^{-x^2}$ and $g : x \longrightarrow e^{-x}$ find the corresponding angle of intersection.

At a point of intersection of the two graphs,

$$e^{-x^2} = e^{-x} ;$$

hence

$$e^{-x^2} e^x = e^{x(1-x)} = 1 ,$$

which has the solutions $x = 0$ and $x = 1$.

Since $f'(x) = -2xe^{-x^2}$, $g'(x) = -e^{-x}$, at $x = 0$ the graphs have the slopes 0 and -1 , respectively; and at $x = 1$, the slopes $-\frac{2}{e}$ and $-\frac{1}{e}$, respectively. If m_1 denotes the slope of $y = f(x)$ and m_2 the slope of $y = g(x)$ at a point of intersection then the angle between the graphs is

$$\phi = \arctan \frac{m_1 - m_2}{1 + m_1 m_2} .$$

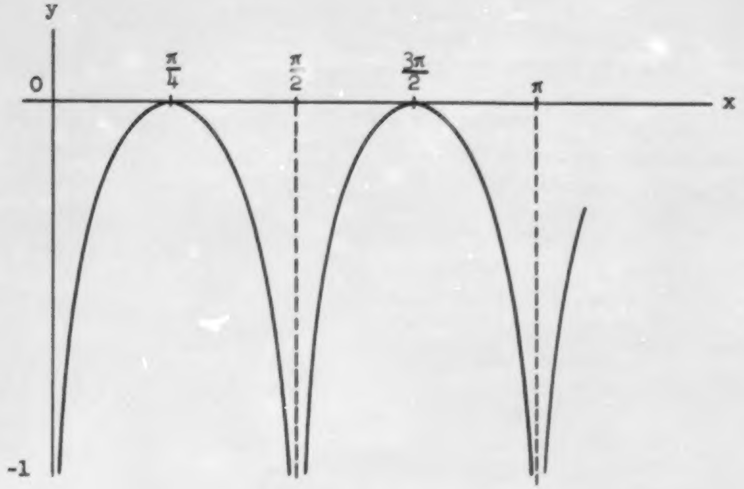
At $x = 0$, then, $\phi = \arctan 1 = \frac{\pi}{4}$.

At $x = 1$, $\phi = \arctan \frac{-e}{e^2 + 2}$.

11. Show that the curve $y = \log|\sin 2x|$ is everywhere flexed downward. Sketch the curve.

$$y' = \frac{2\cos 2x}{\sin 2x}, \quad y'' = \frac{-4}{\sin^2 2x}.$$

Since $y'' < 0$ for all x in the domain of $f : x \longrightarrow \log|\sin 2x|$, the curve is everywhere flexed downward. The function f is periodic with period $\frac{\pi}{2}$. The graph is symmetric about the y-axis and the line $x = \frac{\pi}{4}$; it has vertical asympotes at $x = \frac{n\pi}{2}$ for integral n; maximum $y = 0$ at $x = \frac{\pi}{4} + \frac{n\pi}{2}$.



$f : x \longrightarrow \log|\sin 2x|$

12. Find the area of the standard region under the graph of f over the given interval

(a) $f : x \longrightarrow e^x$, $[-1,1]$; $\int_{-1}^{1} e^x\,dx = e - \frac{1}{e}$.

(b) $f : x \longrightarrow e^{1-x}$, $[-1,1]$; $\int_{-1}^{1} e^{1-x}dx = -e^{1-x}\Big|_{-1}^{1} = e^2 - 1$.

(c) $f : x \longrightarrow \tan x$, $[0, \frac{\pi}{3}]$;

$$\int_0^{\pi/3} \tan x\,dx = \int_0^{\pi/3} \frac{\sin x}{\cos x}\,dx = -\int_0^{\pi/3} \frac{d(\cos x)}{\cos x}$$

$$= -\log \cos x \Big|_0^{\pi/3} = -\log \frac{1}{2} + \log 1$$

$$= \log 2 \ .$$

13. Find the volume of the solid of revolution generated by f on the given interval.

(a) $f : x \longrightarrow e^x$, $[-1,1]$; $\pi \int_{-1}^{1} e^{2x}\,dx = \frac{\pi}{2}(e^2 - e^{-2})$.

(b) $f : x \longrightarrow e^{1-x}$, $[-1,1]$; $\pi \int_{-1}^{1} e^{2(1-x)}dx = -\frac{\pi}{2}\,e^{2(1-x)}\Big|_{-1}^{1}$

$$= \frac{\pi}{2}(e^4 - 1) \ .$$

14. Derive a formula for the volume V of the solid of revolution generated by $f : x \longrightarrow a^x$ $(a > 0)$ on $[0,b]$.

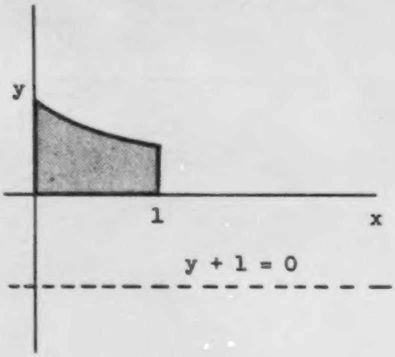
$$V = \pi \int_0^b a^{2x}dx$$

$$= \pi \int_0^b e^{2x \log a}dx$$

$$= \frac{\pi e^{2x \log a}}{2 \log a}\Bigg|_0^b = \frac{\pi}{2 \log a}(e^{2b \log a} - 1)$$

$$= \frac{\pi}{2 \log a}(a^{2b} - 1) \ .$$

15. Find the volume of the solid generated by revolving the region bounded by the curve $y = e^{-x/2}$, the x-axis, the y-axis and the line $x = 1$ about the line $y + 1 = 0$.



$$V = \pi \int_0^1 [(e^{-x/2} + 1)^2 - 1]dx$$

$$= \pi \int_0^1 [e^{-x} + 2e^{-x/2}]dx$$

$$= \pi[-e^{-x} - 4e^{-x/2}]\Big|_0^1$$

$$= \pi[5 - \frac{1}{e} - \frac{4}{\sqrt{e}}] .$$

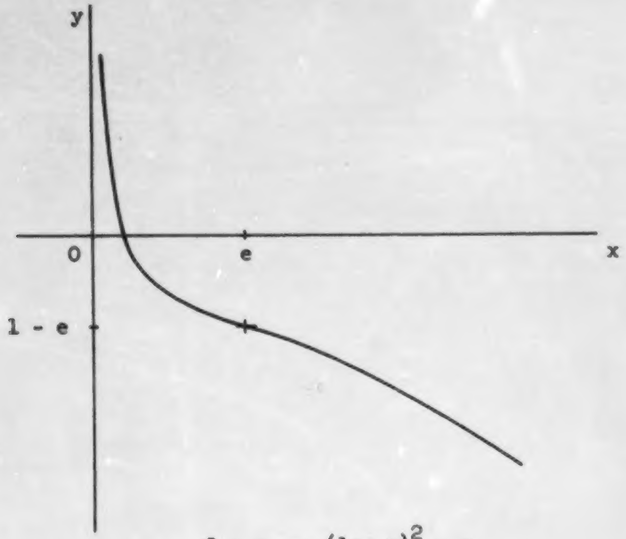
16. Sketch the graphs of each of the following, indicating extrema, points of inflection, intervals of downward and upward flexure.

(a) $f : x \longrightarrow (\log x)^2 - x$

$$f'(x) = \frac{2 \log x}{x} - 1 = \frac{2 \log x - x}{x}$$

$$f''(x) = \frac{x(\frac{2}{x} - 1) - (2 \log x - x)}{x^2} = \frac{2(1 - \log x)}{x^2} .$$

To investigate the zeros of f', consider $u = 2 \log x - x$. Since $u' = \frac{2 - x}{x}$, and $u'' = -\frac{2}{x^2} < 0$ for $x > 0$, the maximum of u occurs at $x = 2$; since $2 \log 2 - 2 < 0$ it follows that $f'(x)$ is everywhere negative and f has no extrema. There is a point of inflection at $(e, 1-e)$: the graph is flexed upward for $x < e$ and flexed downward for $x > e$.



$f : x \longrightarrow (\log x)^2 - x$

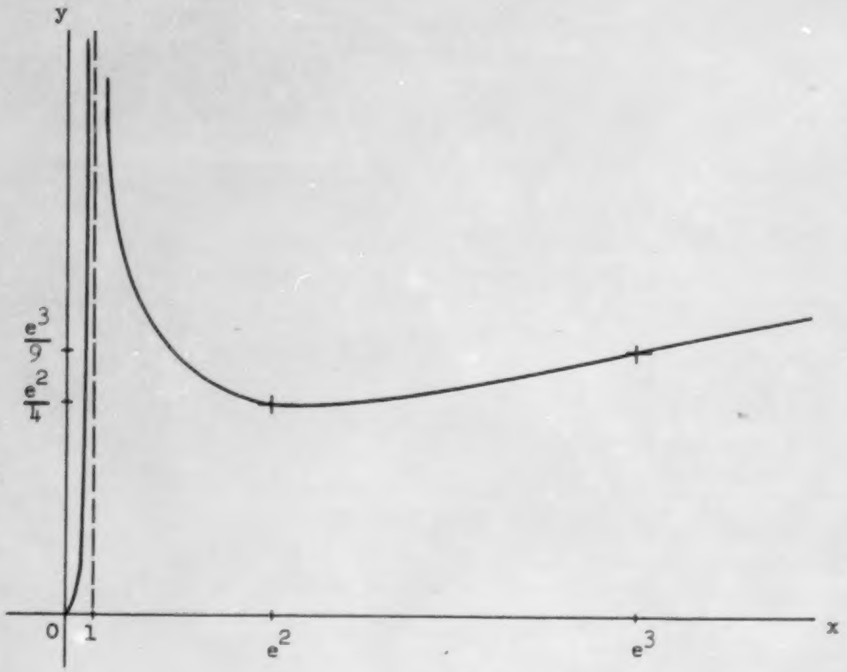
(b) $f : x \longrightarrow \dfrac{x}{(\log x)^2}$

$f'(x) = \dfrac{\log x - 2}{(\log x)^3}$ $\qquad$ $f''(x) = \dfrac{2(3 - \log x)}{x(\log x)^4}$

Point of inflection $\left(e^3, \dfrac{e^3}{9}\right)$

The graph is flexed upward for $x < e^3$, downward for $x > e^3$.

Local minimum $\dfrac{e^2}{4}$ at $x = e^2$.



$$f : x \longrightarrow \frac{x}{(\log x)^2}$$

17. If A is a constant evaluate $\int_{\cosh^2 x}^{\sinh^2 x} A\,dt$.

Answer: $-A$

18. Integrate x^2e^{mx} by assuming the integral has the form $(ax^2 + bx + c)e^{mx} + k$. Generalize this result.

Set $D[(ax^2 + bx + c)e^{mx}] = x^2e^{mx}$ and equate coefficients of like powers in the polynomial factors:

$$e^{mx}[(2ax + b) + m(ax^2 + bx + c)] = e^{mx}[ma\ x^2 + (2a + mb)x + (b + me)] = x^2e^{mx};$$

whence,

$$ma = 1\ ,\ 2a + mb = 0\ ,\ b + mc = 0$$

and then

$$a = \frac{1}{m} , \; b = -\frac{2}{m^2} , \; c = \frac{2}{m^3} ;$$

that is,

$$\int x^2 e^{mx} dx = (\frac{x^2}{m} - \frac{2x}{m^2} + \frac{2}{m^3})e^{mx} + k .$$

In general, set

$$\int x^n e^{mx} dx = (a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0)e^{mx} + C ;$$

then

$$x^n e^{mx} = e^{mx}\left[m \sum_{k=0}^{n} a_k x^k + \sum_{j=1}^{n} j \, a_j \, x^{j-1}\right]$$

$$= e^{nx}\left[m \sum_{k=0}^{n} a_k x^k + \sum_{k=0}^{n-1} (k + 1)a_{k+1} x^k\right]$$

$$= e^{mx}\left\{ma_n x^n + \sum_{k=0}^{n-1} x^k[ma_k + (k + 1)a_{k+1}]\right\} ;$$

hence,

$$ma_n = 1 , \; ma_k + (k + 1)a_{k+1} = 0 , \quad (k = 0,1,2,\ldots,n-1) .$$

Consequently,

$$a_n = \frac{1}{m} , \; a_{n-1} = -\frac{n}{m^2} , \; a_{n-2} = \frac{n(n - 1)}{m^3} , \; \ldots ,$$

$$a_k = \frac{n! \, (-1)^{n-k}}{k! \, m^{n-k+1}} , \; \ldots , \; a_0 = \frac{n!(-1)^n}{m^{n+1}} ;$$

hence,

$$\int x^n e^{mx} dx = \frac{n!(-1)^n}{m^{n+1}}\left[\frac{(-mx)^n}{n!} + \frac{(-mx)^{n-1}}{(n - 1)!} + \ldots + \frac{(-mx)^k}{k!} + \ldots + 1\right]e^{mx} + C .$$

19. (a) Verify that the derivative of

$$f(t) = (a \cos \omega t + b \sin \omega t)e^{kt}$$

is an expression of the same general form.

Answer: $f'(t) = [(ak + b\omega)\cos \omega t + (bk - a\omega)\sin \omega t]e^{kt}$.

(b) Integrate $e^{ax}\sin bx$.

Take $f'(x) = e^{ax}\sin bx$ in the result of Part (a); i.e., set

$$f(x) = (\alpha\cos bx + \beta\sin bx)e^{ax}$$

(i) $f'(x) = [(\alpha a + \beta b)\cos bx + (\beta a - \alpha b)\sin bx]e^{ax}$

$= e^{ax}\sin bx.$

Equate coefficients of like terms to obtain

$$\alpha a + \beta b = 0\ ,\ \beta a - \alpha b = 1\ ,$$

which yields for α and β ,

$$\alpha = \frac{-b}{a^2 + b^2}\ ,\ \beta = \frac{a}{a^2 + b^2}\ ,$$

and for the integral,

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}[a\sin bx - b\cos bx] + C\ .$$

(c) Integrate $e^{ax}\cos bx$.

Proceed as in Part (b), with $e^{ax}\cos bx$ in place of $e^{ax}\sin bx$ on the right side of Equation (i). Alternatively, note that

$$\int e^{ax}\cos bx\,dx = e^{\frac{-\pi a}{2b}}\int e^{a(x + \frac{\pi}{2b})}\sin b(x + \tfrac{\pi}{2b})d(x + \tfrac{\pi}{2b})$$

$$= \frac{e^{ax}}{a^2 + b^2}[a\sin(bx + \tfrac{\pi}{2}) - b\cos(bx + \tfrac{\pi}{2})] + C$$

$$= \frac{e^{ax}}{a^2 + b^2}[a\cos bx + b\sin bx] + C\ .$$

(d) Show that $f(t)$ in (a) is a solution of the differential equation

$$D^2f - 2k\,Df + (k^2 + \omega^2)f = 0\ .$$

Enter $f''(t) = e^{kt}[(ak^2 + 2b\omega k - a\omega^2)\cos\omega t + (bk^2 - 2a\omega k - b\omega^2)\sin\omega t]$ and the expressions for $f'(t)$ and $f(t)$ in the differential equation.

0. Establish the following limits.

(a) $\lim\limits_{x\sim 0} x^m \log(1 + x) = 0 \text{ , } m > -1 \text{ .}$

Observe first that

$$\lim_{x\sim 0} \frac{\log(1 + x)}{x} = \lim_{x\sim 0} \frac{\log(1 + x) - \log 1}{x}$$

$$= D \log z\Big|_{z=1} = \frac{1}{z}\Big|_{z=1} = 1 \text{ .}$$

Since $m + 1 > 0$,

$$\lim_{x\sim\infty} x^m \log(1 + x) = (\lim_{x\sim 0} x^{m+1}) (\lim_{x\sim 0} \frac{\log(1 + x)}{x})$$

$$= 0 \cdot 1 = 0 \text{ .}$$

(b) $\lim\limits_{x\sim\infty} x^m \log(1 + x) = 0 \text{ , } m < 0 \text{ .}$

Use Lemma 8-3 as follows

$$\lim_{x\sim\infty} x^m \log(1 + x) = \lim_{x\sim\infty} x^m \log x + (\lim_{x\sim\infty} x^m)(\lim_{x\sim\infty} \log[1 + \frac{1}{x}])$$

$$= 0 + 0 \cdot 0 = 0 \text{ .}$$

(c) $\lim\limits_{x\sim 0} x^p \log x = 0 \text{ , } p > 0 \text{ .}$

Take $x = \frac{1}{z}$; then

$$\lim_{x\sim 0} x^p \log x = -\lim_{z\sim\infty} \frac{\log z}{z^p} \text{ .}$$

The result follows by Lemma 8-3.

21. Show

$$\lim_{x\sim a} \phi(x)^{\psi(x)} = L , \qquad L > 0 ,$$

if and only if

$$\lim_{x\sim a} \psi(x) \log \phi(x) = \log L .$$

If $\lim_{x\sim a} \phi(x)^{\psi(x)} = L$ then by the continuity of log ,

$$\log L = \log \lim_{x\sim a} \phi(x)^{\psi(x)} = \lim_{x\sim a} \log \phi(x)^{\psi(x)}$$

$$= \lim_{x\sim a} \psi(x) \log \phi(x) .$$

Conversely, if $\lim_{x\sim a} \psi(x) \log \phi(x) = \log L$ then by the continuity of exp ,

$$L = \exp\{\log L\} = \exp\{\lim_{x\sim a} \psi(x)\log \phi(x)\}$$

$$= \lim_{x\sim a} \exp\{\psi(x)\log \phi(x)\} = \lim_{x\sim a} \exp\{\log \phi(x)^{\psi(x)}\}$$

$$= \lim_{x\sim a} \phi(x)^{\psi(x)} .$$

22. Using the previous exercise, evaluate each of the limits in Exercises 8-6, Number 4.

Let L represent the indicated limit in each case.

(a)

$$\log L = \log \lim_{x\sim\infty} (1 + \frac{2}{x})^x$$

$$= \lim_{x\sim\infty} x \log (1 + \frac{2}{x})$$

$$= \lim_{z\sim\infty} \frac{2 \log (1 + z)}{z} \qquad (z = \frac{2}{x})$$

$$= 2$$

where the last step employs $\lim_{z\sim 0} \frac{\log(1 + z)}{z} = 1$ (proved in No. 20(a)). It follows that $L = e^2$.

(b) $\log \lim_{x\sim\infty} (1 + \frac{1}{2x})^{x/3} = \frac{1}{6} \lim_{z\sim\infty} \frac{\log(1 + z)}{z}$, $\qquad (z = \frac{1}{2x})$;

hence, $\qquad L = e^{1/6}$.

(c) $$\log \lim_{h \to 0} (1 + 2h)^{3/h} = 6 \lim_{x \to 0} \frac{\log(1 + x)}{x} , \qquad (x = 2h) ;$$

$$L = e^{6} .$$

(d) $$\log \lim_{h \to 0} (1 + \sin h)^{2 \csc h} = 2 \lim_{z \to 0} \frac{\log (1 + z)}{z} , \qquad (z = \sin h) ;$$

$$L = e^{2} .$$

(e) $$\log \lim_{h \to 0} (1 - \frac{h}{2})^{1/2h} = - \frac{1}{4} \log \frac{(1 + z)}{z} , \qquad (z = - \frac{h}{2}) ;$$

$$L = e^{-1/4} .$$

23. Evaluate each of the following limits if it exists.

Let L represent the indicated limit in each case,

$$L = \lim_{x \to 0} (\frac{\sin x}{x})^{1/x^p}$$

and

$$\log L = \lim_{x \to 0} \frac{1}{x^p} \log (\frac{\sin x}{x}) .$$

Now use the estimates for $\sin x$ obtained in Section 7-5:

$$x - \frac{x^3}{6} \le \sin x \le x - \frac{x^3}{6} + \frac{x^5}{120} , \qquad (0 < x < \frac{\pi}{2}) ;$$

It follows that

$$1 - \frac{x^2}{6} \le \frac{\sin x}{x} \le 1 - \frac{x^2}{6} + \frac{x^4}{120} , \qquad (0 < x < \frac{\pi}{2}) ;$$

the inequality holds also for $x < 0$ since the functions involved are all even. Now, from Exercises 8-2, Number 6(a),(b),

$$-t - \frac{t^2}{2(1 - t)} \le \log(1 - t) \le - t \qquad (0 \le t \le 1) .$$

For simplicity restrict $t \le \frac{1}{2}$, so that

$$-t - t^2 \le \log (1 - t) \le -t .$$

Combine the preceding results to obtain

$$-\frac{x^2}{6} - \frac{x^4}{36} \leq \log\left(\frac{\sin x}{x}\right) \leq -\frac{x^2}{6} + \frac{x^4}{120}$$

when $\frac{x^2}{6} < \frac{1}{2}$. Now apply the Squeeze Theorem to obtain

(a) $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x} = e^0 = 1$,

(b) $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}$.

Furthermore, from the same inequality

(c) $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x^3}$ does not exist.

24. The formula for the normal probability curve used in statistics is

$$y = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2}$$

where m is the average (or mean) value of x and σ is the standard deviation and measures the spread of the curve. Find the extremum and points of inflection and sketch the curve. For simplicity let $m = 0$ and $\sigma = 1$ so that

$$y = f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} .$$

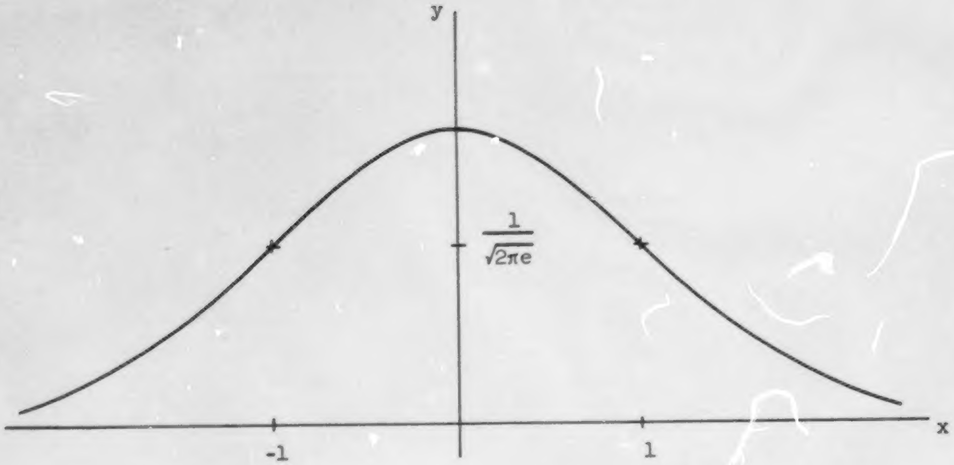
Since

$$f'(x) = -\frac{x}{\sqrt{2\pi}} e^{-x^2/2}$$

and

$$f''(x) = \frac{x^2 - 1}{\sqrt{2\pi}} e^{-x^2/2} ,$$

the graph $y = f(x)$ has inflection points at $\left(\pm 1, \frac{1}{\sqrt{2\pi e}}\right)$; the graph is flexed upward for $|x| > 1$, downward for $|x| < 1$; it has a maximum $y = \frac{1}{\sqrt{2\pi}}$ at $x = 0$. The x-axis is a horizontal asymptote.



$$f : x \longrightarrow \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$$

25. Determine the rectangle of largest area with base along the x-axis and two vertices on the graph of $f : x \longrightarrow e^{-x^2}$.

By symmetry, the vertices may be written $(\pm x, 0)$ $(\pm x, e^{-x^2})$. The area of such a rectangle is $A = 2xe^{-x^2}$. The derivative $\frac{dA}{dx} = (2 - 4x^2)e^{-x^2}$ has zeros at $x = \pm \frac{1}{\sqrt{2}}$ corresponding to the desired maximum.

26. Prove that the function f given by

$$f(x) = \begin{cases} \frac{\sin x}{x} & , x \neq 0 \\ 1 & , x = 0 \end{cases}$$

(a) is continuous for all x .
(b) is differentiable for all x .

The function is differentiable for $x \neq 0$; it is necessary to prove only that it is differentiable at $x = 0$

Set

$$r(x) = \frac{f(x) - f(0)}{x} = \frac{\frac{\sin x}{x} - 1}{x} , \qquad (x \neq 0) .$$

From the estimates for $\frac{\sin x}{x}$ of Number 23,

$$|r(x)| = \frac{1}{|x|}\left|\frac{\sin x}{x} - 1\right| \leq \frac{x^2}{6|x|} \leq \frac{|x|}{6}$$

(provided $\frac{x^4}{120} < \frac{x^2}{6}$, which is certainly true if $|x| < 1$).

Consequently, $\lim_{x \to 0} r(x) = f'(0) = 0$; thus $f'(0)$ exists.

Teacher's Commentary

Chapter 9

GROWTH, DECAY AND COMPETITION

Our main purpose in this chapter is to show how one mathematical model can serve to link phenomena and processes which occur in all sciences.

In the preceding chapters, in the course of a relatively systematic development of fundamental theory, we stressed the novel features of calculus and indicated its relations to arithmetic, algebra, and geometry. We mentioned some of the classes of problems for which calculus provides the right language, the right tools, and even the right reflexes, but up to the present stage the applications we considered were intended to motivate the development of the subject and to illustrate some of its special consequences. But now we have available a complete enough "tool kit" for more extensive applied work. Pushing this analogy, we face the following choice: should we use the tools merely for a collection of different exercises designed to show which tools can be used for what projects; or should we attempt some systematic development of applications of the calculus to the sciences. To exhibit applied mathematics as a set of cook book recipes would be criminal. We want to avoid serving up the mathematical sciences as the traditional stew of unrelated ingredients. There are much better approaches. Let us attempt to show how the master chefs and gourmets perform, or at least how they systematize their work after it has been done on a catch-as-catch-can basis.

There are two different systematic approaches we could follow. We could pick one general method of the calculus and show how it is used in many different sciences, or we could pick one general topic of science and show how different methods of the calculus have furthered its development. Because these two procedures are essentially different, and because both are indicative of the way applied mathematicians and mathematical scientists work in practice, we illustrate both procedures.

In the present chapter we follow the first approach. We develop mathematical models for growth, decay, and competition, and show how these models are used in physics, biology, chemistry, psychology, and sociology. To stress the generality of the mathematics and of the fact that the model is completely independent of any scientific discipline, we introduce the equations as a model for the spreading of stories. The model for story-spreading picks up the threads of Section 1-3 on "How did you hear about calculus? How did you hear about Helen of Troy? ..." etc. When we convert the equations to a model for forgetting and memorizing isolated facts, we pick up other threads of Section 1-3. About half of the applications mentioned in Section 1-3 are covered in this chapter, and the rest are either covered in Chapter 15 (Geometrical Optics and Waves) or illustrated elsewhere in the text.

While this is the first of two chapters (the second being Chapter 15) on the relation of mathematics to science, we do not attempt to teach science. Our aim is merely to illustrate how mathematics (particularly the calculus through the transcendental functions presented in Chapter 8) is applied to the sciences.

We say, "Read the story through at one sitting." We wish to encourage the student to follow the central theme, to view the forest before focusing upon the individual trees. We urge that exercises not be assigned until the student has and the opportunity to appreciate the whole.

Many interesting problems which arise naturally from the discussions of this chapter (applications requiring the solution of differential equations) cannot be handled until certain techniques are developed in Chapter 10. Strictly speaking, we need the method of integration by parts (Section 10-4) to get Equation (15), in Section 9-3(iv); but that

$$\int_0^t e^{At} \sin \omega t \, dt = \frac{-\omega e^{At} \cos \omega t + A e^{At} \sin \omega t}{\omega^2 + At}$$

is true, can be made plausible as early as Chapter 8 (Miscellaneous Exercises, No. 19).

An applied problem of this chapter frequently becomes for us the problem of integrating a rational function. We should keep in mind (in anticipation of Section 10-5) that for a rational function there always exists a formal integral in terms of elementary functions.

We note that the mature areas of today's sciences have passed through a long historical development culminating in their present systematic deductive stage. There have been stages of observation and classification of superficially related phenomena, of measurement and the collection of data, of seeking detailed relations, of isolating essential factors, of creating new concepts to link the more obscure relations, of seeking laws of nature which summarize and generalize observations, of testing such laws to determine the class of phenomena to which they apply, of inventing more general more abstract laws, and of predicting less obvious phenomena. Cycles of observation, speculation, and verification mark our attempts to systemize our knowledge of nature. We exploit yesterday's science to invent today's tools to probe the science of tomorrow.

In scientific usage there are often notational ambiguities which we have not eliminated. When we say N is a function of t (footnote in Section 9-3) we mean that there exists a function $\nu : t \longrightarrow N$, not that N itself is a function. Careless paraphrase of this expression leads to the use of $N(t)$ for $\nu(t)$, a particularly bad but common notation in science. While at first encounter, such ambiguities may distress the student, he may take comfort in the realization that comprehension of this and future scientific material we may read can be enhanced by less cumbersome -- if less precise -- notation.

We have not tracked down original references to Verhulst's work on population growth. Our account (9-4(ii)) is based on Max Schiffer's lively discussion of population growth in Applied Mathematics in the High School (SMSG Studies in Mathematics X, 1963). Schiffer assumes no calculus and is forced to used difference equations instead of differential equations.

There are just three sets of exercises, namely Exercises 9-2, 9-3, and 9-5, placed at the conclusion of the chapter for the reasons mentioned in the text.

Solutions Exercises 9-2

1. A colony of bacteria grows at a rate proportional to its population. Initially there are 2,000 bacteria and one day later, 5,000. What should the population of the colony be after 5 days if it grows unrestrictedly?

By the law of unregulated growth $N = N_0e^{At}$. If t is given in days, then for $t = 1$,

$$5000 = 2000\, e^{A},$$

whence

$$A = \log \frac{5}{2}.$$

Thus at $t = 5$,

$$N = 2000\, e^{5 \log \frac{5}{2}} = 2000\left(\frac{5}{2}\right)^5$$

$$= \frac{5^8}{2} \approx 1.95 \times 10^6 .$$

2. The rate of production of a given chemical in a given reaction increases by 2% for each degree of increase in temperature. What increase in temperature is needed to double the production rate?

Implicit in the statement of the problem is that if r is the production rate and T the temperature, then

$$\frac{dr}{dt} = ar .$$

Consequently, $r = r_0 e^{at}$ and $1.02r = r_0 e^{a(T+1)}$; hence $a = \log 1.02$. To double production, the increase ΔT in temperature must satisfy

$$2r = r_0 \, e^{a(T+\Delta T)}$$

whence,

$$\Delta T = \frac{\log 2}{a} = \frac{\log 2}{\log(1.02)} .$$

From $\log 2 \approx 0.693$, $\log(1.02) \approx 0.02$,

$$\Delta T \approx 34.6 \text{ degrees.}$$

3. Show for any solution of Equation (3) that over a fixed time period τ the change in N is in constant proportion to the value of N at the beginning of the period, independently of the initial time t .

The problem is to prove for any solution $N(t)$ of

$$\frac{dN(t)}{dt} = A\,N(t)$$

that

$$N(t + \tau) - N(t) = K\,N(t)$$

where K is constant. Any solution of the differential equation can be put in the form $N(t) = N_0 e^{At}$; whence,

$$\begin{aligned} N(t + \tau) - N(t) &= N_0 e^{A(t+\tau)} - N_0 e^{At} \\ &= (e^{A\tau} - 1) N_0 e^{At} = K\,N(t) \end{aligned}$$

where

$$K = e^{A\tau} - 1 .$$

4. Write the differential equation for the growth of a bacterial population which increases 2.5% every hour. If there are N_0 bacteria at the start, how many are there at the end of 10 hours?

The differential equation is $N'(t) = A\,N(t)$ where A is given by $1.025\,N_0 = N_0 e^A$; i.e., $A = \log 1.025$. At the end of 10 hours

$$N(10) = N_0 e^{10 \log 1.025} .$$

Since, for small x ,

$$\log (1 + x) \approx x ,$$

$$N(10) \approx N_0 e^{1/4} \approx 1.3 N_0 .$$

5. The population of a city has been growing at a rate proportional to itself. If the population is now 40,000 and 25 years ago it was 15,000 , find the anticipated population 10 years hence.

Let $t = 0$ indicate the present time. The population $N(t)$ is then known to be given by

$$N(t) = 40{,}000\, e^{At}$$

for $t < 0$ and the same formula is used to forecast. The constant A is determined by

$$N(-25) = 40{,}000 e^{-25A} = 15{,}000$$

which yields

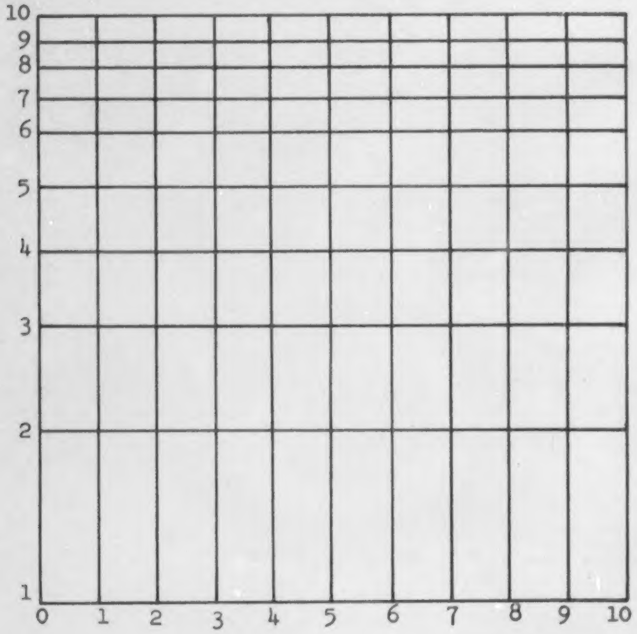
$$A = \frac{1}{25} \log \left(\frac{8}{3}\right).$$

The anticipated population in 10 years is then

$$N(10) = 40{,}000 e^{10A} = 40{,}000 \left(\frac{8}{3}\right)^{2/5} ,$$

approximately 58,000 .

6. Semi-logarithmic coordinates represent the ordinate on a logarithmic scale and the abscissa on a uniform scale (see figure).



(a) The graph of $y = Ae^{bx}$ is given by a straight line on semilogarithmic paper. Why?

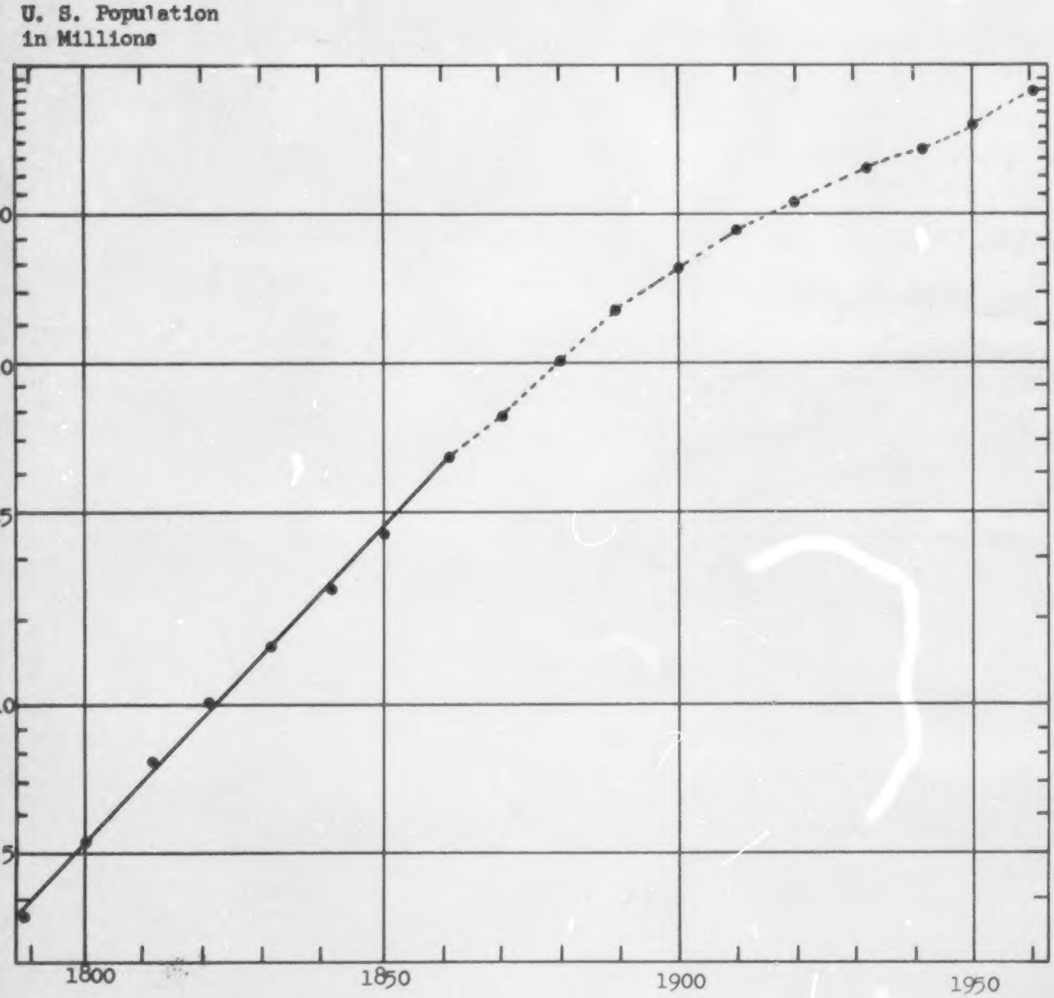
Let z be a conventional ordinate with a uniform scale; then $z = C \log y$ where C is positive and constant. The graph $y = Ae^{bx}$ in semi-logarithmic coordinates is described in cartesian coordinates by

$$z = C \log y = C \log A + Cbx ,$$

a linear equation.

(b) The census figures for the total population of the United States obtained at ten-year intervals from 1790 to 1960 are given in millions as follows:

3.93 , 5.31 , 7.24 , 9.64 , 12.9 , 17.1 , 23.2 , 31.4 , 38.6 , 50.2 , 62.9 , 76.0 , 92.0 , 106 , 123 , 132 , 151 , 179 .
Plot the population vs. time on semilogarithmic graph paper. Over what time intervals do the points appear to lie on a straight line? Use Formula (1) to obtain a reasonable average value of b for each of these periods.



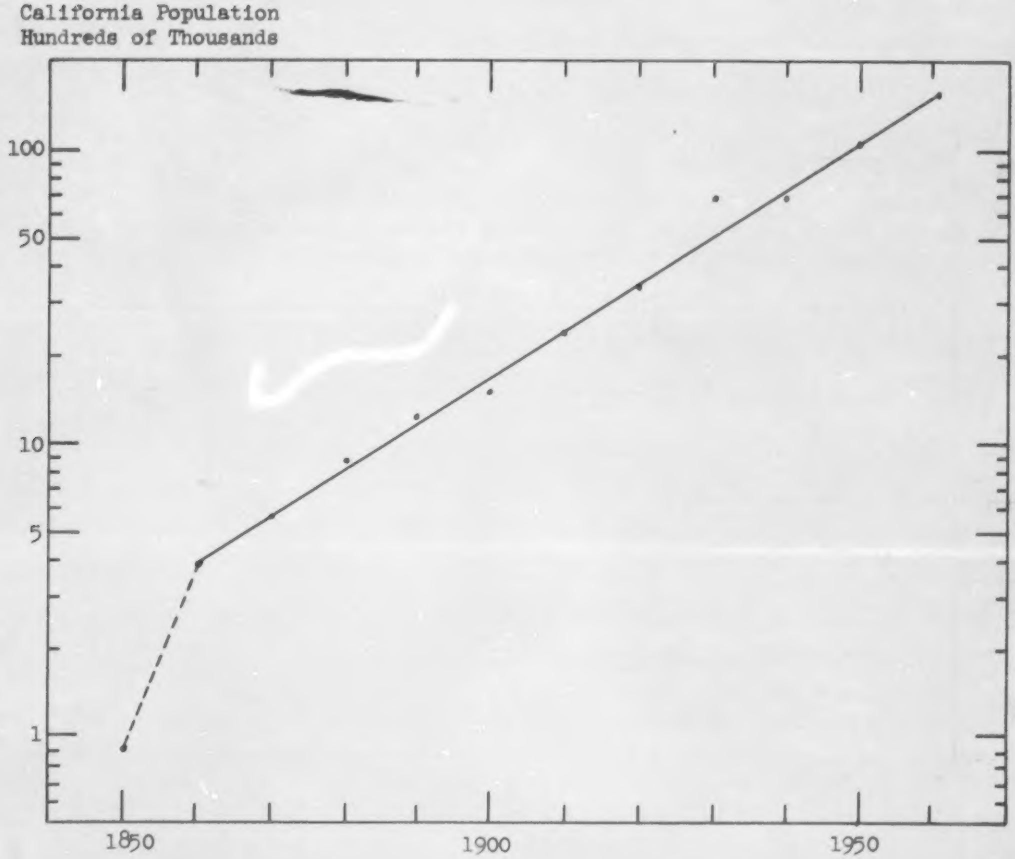
The graph is very nearly linear for the period 1790 to 1860. If the graph is linear from time t_0 to t_1 then

$$b = \frac{\log y_1 - \log y_0}{t_1 - t_0} = \frac{1}{t_1 - t_0} \log \frac{y_1}{y_0} .$$

Take $t_0 = 1790$, $t_1 = 1860$ to obtain

$$b = \frac{1}{70} \log \frac{31.4}{3.93} \approx \frac{\log 8}{70} \approx \frac{3 \times 0.693}{70} \approx .03 .$$

(c) Do the same for the census figures for the state of California for the period 1850 to 1960 (Figures in hundreds of thousands): 0.926, 3.80, 5.60, 8.65, 12.1, 14.9, 23.8, 34.3, 56.8, 69.1, 106, 157.



The graph is approximately linear from $t_0 = 1860$ to $t_1 = 1960$; accordingly

$$b = \frac{1}{t_1 - t_0} \log \frac{y_1}{y_0} \approx \frac{1}{100} \log \frac{157}{0.926} \approx \frac{1}{100} \log 170 \approx .051 \ .$$

(d) Look up the figures for your own state in an almanac and study its population growth in the same fashion.

Solutions Exercises 9-3

1. If the half-life τ of a radioactive substance is given in seconds, show that the fraction of the substance decaying in one second is approximately $\frac{\log 2}{\tau}$. (Hint: Assume that τ is a large number.)

The fraction of the substance decaying in time t is

$$\frac{N_0 - N(t)}{N_0} = 1 - e^{-At}$$

where $A = \frac{\log 2}{\tau}$. In one second, the fraction decaying is approximately

$$1 - e^{-A} \approx 1 - (1 - A) = A$$

with an error at most $\frac{A^2}{2}$ (from the result of Exercises 8-6, No. 1(b).

2. Verify that the half-life of a radioactive substance is independent of the initial time and the initial amount of the substance.

The decay coefficient A in the differential equation $\frac{dN(t)}{dt} = -AN(t)$ is assumed to be constant. Whatever the initial amount N_0 of the substance, the half-life τ is defined by

$$\frac{N(\tau)}{N_0} = e^{-A\tau} = \frac{1}{2} ;$$

hence $\tau = \frac{\log 2}{A}$ which is independent of N_0 . A change in the initial time amounts to a change of the zero in the time scale, hence a change of the initial amount of substance. Hence, indepence of initial time follows from indepence of N_0 .

3. A 5000 cubic foot garage containing a high concentration of carbon monoxide is being flushed out by an air pump whose capacity is 1000 ft^3/min. Assuming that the air mixture in the garage remains uniform (perfect mixing), determine how long it takes for the concentration of carbon monoxide to fall to $\frac{1}{10}$ its initial value.

The concentration C is assumed to be reduced by replenishment of the exhausted air by completely fresh air.

$$\frac{dC}{dt} = -Cr$$

where r is the proportion of the total volume per unit time which is replaced by fresh air. In this case, $r = \frac{1}{5}$, hence

$$C = C_0 e^{-t/5}$$

where C_0 is the initial concentration. For $\frac{C}{C_0} = \frac{1}{10}$, $e^{t/5} = 10$;

hence,

$$t = 5 \log 10 \approx 11.5 \text{ mins.}$$

4. For a small body in air or liquid, the rate of heat loss is approximately proportional to the difference in temperature between the body and the surrounding medium.
A thermometer which registers 72^o indoors is taken outdoors where the temperature is 12^o . One minute later the thermometer registers 42^o .
(a) Obtain a formula for the thermometer reading r at any time t .

Let x be the difference in temperature between the thermometer and its surroundings; then

$$\frac{dx}{dt} = -Ax$$

where A is to be determined. From the given data, for $t = 1$, the decay law $x = x_0 e^{-At}$ yields

$$30 = 60\, e^{-A} .$$

It follows that $A = \log 2$; hence

$$x = \frac{x_0}{2^t} .$$

(b) What is the reading at the end of two minutes?

$x = \frac{60}{4} = 15$. The thermometer reading is $x + 12 = 27$ degrees.

(c) If the thermometer is left outdoors how long does it take to reach a reading of 18° ?

A reading of 18° corresponds to $x = 6$. Then $6 = \frac{60}{2^t}$; hence

$$t = \frac{\log 10}{\log 2} = 3.3 \text{ mins.}$$

(d) On another day, it takes one minute out-of-doors for the thermometer to drop from its indoor reading of 72° to a reading of 32° . Since it is too cold to stay outside and wait for the thermometer to reach an equilibrium reading, calculate the outside temperature.

Let θ_0 be the unknown outside temperature and θ the thermometer reading; then

$$\theta - \theta_0 = (72 - \theta_0)e^{-At} = \frac{72 - \theta_0}{2^t}$$

and

$$\theta_0 = \frac{\theta 2^t - 72}{2^t - 1} .$$

At $t = 1$, $\theta = 32^\circ$; hence

$$\theta_0 = -8^\circ .$$

5. A thermometer which registers 70° indoors is taken outdoors. Five minutes later it registers 65° and ten minutes after it was taken outdoors it registers 62° .

(a) Calculate the outdoor temperature.

With θ and θ_0 as in Number 4(d),

$$\theta - \theta_0 = (70 - \theta_0)e^{-At} .$$

At $t = 5$, $65 - \theta_0 = (70 - \theta_0)e^{-5A}$,

and at $t = 10$, $62 - \theta_0 = (70 - \theta_0)e^{-10A}$.

Combine the results, using $e^{10A} = (e^{5A})^2$ to obtain

$$\frac{70 - \theta_0}{(62 - \theta_0)} = \frac{(70 - \theta_0)^2}{(65 - \theta_0)^2} ;$$

whence

$$\theta_0 = 57.5^0 .$$

(b) Assuming the thermometer remains outdoors (where the temperature is constant) when will it register 51^o ?

Never.

6. A verterinarian about to engage in surgery on a dog estimates that it will take him 45 minutes to complete the procedure. If 20 mg. of sodium pentobarbitol per kilogram of body weight is needed to barely maintain anesthesia, if the half-life of the anesthetic is five hours in dogs, and if the dog weighs 20 kilograms, how much anesthetic should the doctor administer initially to maintain anesthesia over the estimated duration of the operation? At the end of the 45 minute period the doctor realizes that the dog is beginning to emerge from anesthesia and that the surgery will take a half-hour longer to complete. What dosage of pentobarbitol should be administered at that point?

Let $M(t)$ be the number of milligrams of sodium pentobarbitol per kilogram of body weight, M_0 the amount administered initially, then

$$M(t) = M_0 2^{-t/\tau}$$

where τ is the half-life in hours. Under the preliminary estimates M_0 should be chosen so that $M(\frac{3}{4}) = M_0\, 2^{-3/4 \times 1/5} = 20 \times 20$. Hence, approximately, $M_0 = 443$ mg.

At the end of the 45 minute period the supplementary dosage M_1 is determined by

$$(400 + M_1)2^{-1/10} = 400 ;$$

hence

$$M_1 = 400(2^{1/10} - 1) \approx 400 \times 0.072 \approx 29 \text{ mg.}$$

7. Consider a cigarette being used as a filter for some component (say nicotine) in the smoke. For a linear filter, the amount picked up by the filter over a small length is proportional to the concentration C and the length Δx. Show that the filtration equation is the familiar decay equation

$$\frac{dC}{dx} = -kC$$

and that

$$C = C_0 e^{-kx} .$$

Let v be the velocity of passage through the filter and A the cross-sectional area, and suppose v to be constant. The amount of nicotine removed by the thickness Δx per unit time is proportional to the volumetric flow, to C, and to Δx:

$$\Delta N = -kvAC\,\Delta x .$$

In the time $\Delta t = \frac{\Delta x}{v}$ the volume $V = vA\Delta t$ of smoke passes through the length Δx and the amount of nicotine absorbed is $\Delta N \Delta t$. The change in concentration is the change in the amount of nicotine per unit volume, namely,

$$\Delta C = \frac{\Delta N \Delta t}{V} = -kC\,\Delta x ;$$

whence,

$$\frac{\Delta C}{\Delta x} = -kC .$$

The restriction to "small Δx" is to be interpreted to mean that the preceding statement is true in the limit as Δx approaches 0:

$$\frac{dC}{dx} = -kC .$$

It should be understood that the preceding discussion is not a proof of the validity of the differential equation, but a heuristic argument leading to an appropriate mathematical model.

8. Electrical circuits offer several examples of decay processes. The fundamental electrical quantities are charge q, current

(1) $$I = \frac{dq}{dt} ,$$

and voltage or electromotive force V. The electrical circuit components, resistors, condensers, and coils, have electrical properties measured by certain constants, resistance R, capacity C, and inductance L, respectively. The capacity of a condenser is defined as the ratio of the charge on the condenser to the voltage required to produce it:

(2) $$C = \frac{q}{V} .$$

The resistance of a resistor is defined as the ratio of the imposed voltage to the current it produces:

$$R = \frac{V}{I} . \tag{3}$$

In addition, we need to know Kirchhoff's rule, that the sum of the voltages across the elements of a circuit is zero.

(a) If a condenser at voltage V discharges across a resistor, by Kirchhoff's rule the voltage across the resistor is $-V$.

Employ (1), (2), and (3) to obtain a differential equation for V and solve for V as a function of time subject to the condition $V = V_0$ at $t = 0$.

Combine (1), (2), and (3) (but note that V should be replaced by $-V$ in (3)), to obtain

$$I = -\frac{V}{R} = \frac{dq}{dt} = C\frac{dV}{dt} ;$$

whence,

$$\frac{dV}{dt} = -\frac{1}{RC} V .$$

Answer: $V = V_0 e^{-t/RC} = \frac{q_0}{C} e^{-t/RC}$ where V_0 is the initial voltage on the condenser and q_0 the initial charge.

(b) A coil resists change in current in the same way that a resistor opposes the passage of charge. If the voltage V is applied to a coil the current changes at the rate

$$\frac{dI}{dt} = \frac{V}{L} . \tag{4}$$

If we impose an external voltage E (say by means of a battery) upon a circuit consisting of a resistor and a coil, then by Kirchhoff's rule the sum of the voltages is zero; namely, from (2) and (3)

$$E - IR - L\frac{dI}{dt} = 0$$

whence,

$$\frac{dI}{dt} = \frac{E}{L} - \frac{R}{L} I . \tag{5}$$

Solve (5) subject to the initial condition $I = 0$ at $t = 0$. (Hint: Express the solution as the sum of two terms $I = \frac{E}{R} + J$ where $\frac{E}{R}$ is the steady current which would be set up if the coil were not present, and J is a transient term which represents the effect of the coil.)

Enter the suggested expression for I in (5) to obtain

$$\frac{dJ}{dt} = -\frac{R}{L}J$$

so that, in fact, J does decay in time according to the rule

$$J = J_0e^{-Rt/L} = \frac{E}{R}e^{-Rt/L}$$

where we have used $I = 0$ at $t = 0$ to evaluate J_0. From this result we obtain

$$I = \frac{E}{R}(1 - e^{-Rt/L}) .$$

(c) If there is no external voltage and the current has the value $I = I_0$ at $t = 0$, verify that (5) is the equation of a simple decay process and obtain the solution of (5) under these conditions.

No external voltage means $E = 0$, and (5) becomes $\frac{dI}{dt} = -\frac{R}{L}I$ which has the solution

$$I = I_0e^{Rt/L} .$$

(d) Use Kirchhoff's rule to derive the equation

$$\frac{dE}{dt} - \frac{I}{C} - R\frac{dI}{dt} = 0$$

where a source of electromotive force, E, is connected in a series circuit to a resistor, R, and a condenser, C.

Determine I in terms of t if the electromagnetic force E is constant and the current is 0 at $t = 0$.

The sum of the voltages across the separate circuit elements is zero; hence, from (2) and (3),

$$E - \frac{q}{C} - IR = 0 .$$

Differentiate with respect to t, to obtain

$$\frac{1}{C}\frac{dq}{dt} + R\frac{dI}{dt} = 0 ,$$

where the condition that E is constant $(\frac{dE}{dt} = 0)$ is employed. From (1),

$$\frac{dI}{dt} = -\frac{1}{RC}I ;$$

hence,

$$I = I_0 e^{-t/RC} ,$$

where I_0 is the initial current. In this problem $I_0 = 0$ so that I is identically zero.

9. Consider the differential equation

$$\frac{dx}{dt} = a - bx$$

subject to the initial condition

$$x = x_0 \text{ at } t = 0 .$$

(a) Show that the solution is given by

$$x = \frac{a}{b}(1 - e^{-bt}) + x_0 e^{-bt} .$$

As in Number 8(b), set $x = \frac{a}{b} + y$. Then

$$\frac{dy}{dt} = -by ;$$

whence

$$y = y_0 e^{-bt}$$

and

$$(x - \frac{a}{b}) = (x_0 - \frac{a}{b})e^{-bt}$$

where y_0 and x_0 denote the initial values of y and x.

(b) Show that if $x_0 = 0$ then the result has the same form as that of Number 8(b).

If $x_0 = 0$, then $x = \frac{a}{b}(1 - e^{-bt})$ as compared with the solution $I = \frac{E}{R}(1 - e^{-Rt/L})$ of 8(b). Here $\frac{R}{L}$ corresponds to b, $\frac{E}{L}$ to a.

10. (a) Assume that the rate of inversion of raw sugar is proportional to the amount of raw sugar remaining. If after 4 hours, 500 pounds of raw sugar have been reduced to 200 pounds, how much raw sugar will remain at the end of 12 hours?

The amount x of raw sugar at time t is given by

$$x = x_0 e^{-At} = 500e^{-At} .$$

For $t = 4$, A is determined by

$$200 = 500e^{-4A} \; ;$$

i.e., $A = \frac{1}{4} \log \frac{5}{2}$. At the end of 12 hours,

$$x = 500e^{-12A} = 500(\tfrac{5}{2})^{-3} = \frac{500}{125} \times 8 = 32 \text{ lbs.}$$

(b) Consider a chemical reaction in which the velocity of the reaction (or rate of change in the amount of the substance consumed) is proportional to the quantity of the unconsumed substance at that instant. Let x_0 be the quantity of the substance at $t = 0$, x the quantity that has been converted by time t , and y the quantity of the reaction product. Show that

$$y = x_0(1 - e^{-kt})$$

and give k in terms of the amount y_1 of reaction product at time t_1 .

To solve this problem it is necessary to use the principle of conservation of matter: the total mass is unchanged in the reaction:

$$x + y = x_0 \; .$$

(It is assumed that none of the reaction product is present initially.) From $\frac{dx}{dt} = -kx$,

$$x = x_0 \, e^{-kt} \; ;$$

hence, from conservation of matter,

$$y = x_0 - x = x_0(1 - e^{-kt}) \; .$$

For $t = t_1$,

$$y_1 = x_0(1 - e^{-kt_1}) \; ;$$

whence,

$$k = \frac{1}{t_1} \log \frac{x_0}{x_0 - y_1} \; .$$

11. One dollar silver certificates (payable in silver on demand) are being retired from circulation and replaced by Federal Reserve notes (pure paper). Let N be the number of silver certificates and M the number of Federal Reserve notes in circulation at time t and suppose that the two varieties are uniformly mixed together. Suppose that the number of dollar bills passing through the central banks each day is a constant k, and that all silver certificates among these are replaced by Federal Reserve notes. Suppose that the process is initiated at time $t = 0$ when $N = N_0$ and $M = 0$, and that the total number of dollar bills in circulation is held constant. Determine the way N and M depend on time (measured in banking days) and find the number of days it takes to replace half the silver certificates.

The total number of bills in circulation is constant, $N + M = N_0$. The fraction which passes through the banks per day is $\frac{k}{N_0}$, and since the two types are assumed to be uniformly mixed, the number of silver certificates extracted is $\frac{kN}{N_0}$ per day. Thus the rate of decay of silver certificates in circulation is given by

$$\frac{dN}{dt} = -\frac{k}{N_0} N .$$

Consequently,

$$N = N_0 e^{-kt/N_0}$$

and

$$M = N_0 - N = N_0(1 - e^{-kt/N_0}) .$$

The process has the half-life

$$\tau = \frac{N_0 \log 2}{k} .$$

12. The principle of dating organic matter by radioactive carbon content is based on the observation that the ratio of the concentration of radioactive carbon C^{14} to that of ordinary carbon C^{12} in atmospheric carbon dioxide is maintained at a constant level because of continual cosmic ray bombardment. An organism throughout its life takes up carbon in the same proportions, but after death the relative amount of C^{14} decreases because of radioactive decay without replenishment. Let τ be the half-life of C^{14}. Show how to date an ancient timber if the ratio of C^{14} to C^{12} in the specimen is known.

Let ρ_0 be the ratio of C^{14} to C^{12} in the atmosphere and let ρ be the ratio in the specimen. The amount N of C^{14} in the specimen decays by the rule

$$N = N_0 2^{-t/\tau} \; ;$$

the amount M of C^{12} remains constant. Thus,

$$\rho = \frac{N}{M} = \frac{N_0}{M} 2^{-t/\tau} = \rho_0 2^{-t/\tau} \; .$$

Consequently, the age of the specimen is

$$t = \frac{\rho_0 \tau}{\rho \log 2} \; .$$

13. The continuous model for radioactive decay used here replaces the picture of discrete atoms disintegrating at random times. The utility of such a model depends upon the involvement of a large number of atoms in the process. Reflect realistically about the model. Is it true that a mass of radioactive substance can never completely decay? If not, offer a reasonable estimate of the time of complete disappearance of the substance.

The half-life may be thought of as the time in which the probabilities that a single atom has decayed or has not decayed are equal. The decay law $N = N_0 2^{-t/\tau}$ gives the expected number of atoms of the original population remaining at time t. There is no way to estimate the time of complete disappearance in theory, but in practice it is quite reasonable to take the time as that in which N decays to the threshold of physical detectability. Alternatively, the time might be that in which it becomes equally likely that there is no atom of the original population as that there is; or the time could be taken so that the expected number of atoms remaining is $\frac{1}{2}$:

$$\frac{1}{2} = N_0 2^{-t/\tau}$$

and

$$t = \tau\left[1 + \frac{\log N_0}{\log 2}\right] \; .$$

It is easy to see that this estimate is not very sensitive to the arbitrary choice of cut-off point. If the cut-off is chosen at the threshold of physical detectability, $N = n$ where n is much smaller than N_0, then $n = N_0 2^{-t/\tau}$ and $t = \frac{\tau}{\log 2}[\log n + \log N_0]$.

If $\log n$ is neglected by comparison with $\log N_0$, then

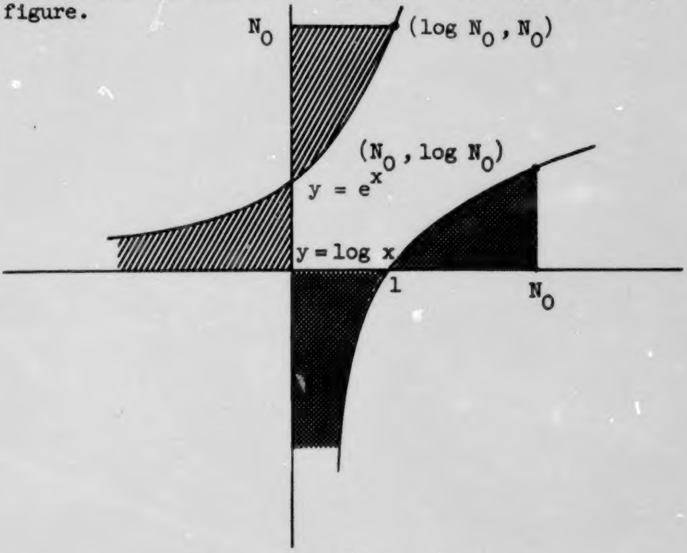
$$t = \frac{\tau \log N_0}{\log 2} = \frac{\log N_0}{A}$$

where A is the decay constant. This, being simple, is as reasonable an estimate as any.

14. Combining Equations (3) and (4) of Section 9-3(1) we obtain for the mean life-time in radioactive decay

$$T = -\frac{1}{AN_0}\int_0^{N_0} \log N \, dN + \frac{1}{A}\log N_0 .$$

The integral $\int_0^{N_0} \log N \, dN$ is the sum of the signed areas of the shaded regions bounded partly by the graph of $y = \log x$ in the following figure.



It is geometrically clear that the areas of the shaded regions are equal to the areas of the hatched regions bounded partly by the graph of $y = e^x$. From this, calculate the integral and hence obtain the mean life-time given in the text.

$$\int_0^{N_0} \log N\, dN = \int_0^1 \log N\, dN + \int_1^{N_0} \log N\, dN$$

$$= -\int_{-\infty}^{0} e^x\, dx + [N_0 \log N_0 - \int_0^{\log N_0} e^x dx]$$

$$= N_0 \log N_0 - \int_{-\infty}^{\log N_0} e^x\, dx .$$

Here the integral over the unbounded interval is defined naturally as $\lim\limits_{a\sim-\infty} \int_a^{\log N_0} e^x\, dx$, if the limit exists; i.e., as

$$\lim_{a\sim-\infty} e^x \Big|_a^{\log N_0} = N_0 - \log_{a\sim-\infty} e^a = N_0 .$$

Enter this result in the expression for the mean life-time to obtain

$$T = -\frac{1}{AN_0}[N_0 \log N_0 - N_0] + \frac{1}{A} \log N_0 = \frac{1}{A} .$$

15. Bacteria and other cells reproduce by splitting in two. What is the average time between "birth" and cell division for a member of the bacterial colony of Exercises 9-2, Number 1?

Define the average time τ between cell divisions as the time which would produce a growth law of the form

$$N = N_0 e^{At}$$

if it were the same for all members of the population. In such an ideal population each member would divide exactly once in any time period $t_0 \le t < t_0 + \tau$. Thus τ is the time it takes the population to double:

$$N(t_0 + \tau) = N_0 e^{A(t_0+\tau)} = 2N_0 e^{At_0} .$$

Consequently, $\tau = \frac{\log 2}{A}$.

For the population of Exercises 9-2, Number 1, we have $A = \log \frac{5}{2}$; hence $\tau = \log 2/\log \frac{5}{2} \approx .756$ days.

16. Here is an example of decay which is not exponential. A spherical moth ball in a closet evaporates away at a rate proportional to its surface area. If half of it (in weight) evaporates away in 10 days, determine how many more days it takes so that one quarter of the original amount is left.

Let M be the mass of the moth ball, R its radius. Assume M is proportional to the volume $V = \frac{4}{3}\pi r^3$. Then,

$$\frac{dV}{dt} = -k4\pi r^2 \ ;$$

whence

$$4\pi r^2 \frac{dr}{dt} = -k4\pi r^2$$

and

$$\frac{dr}{dt} = -k \ .$$

It follows that r decreases linearly,

$$r = r_0 - kt$$

and, in contrast to exponential decay, the entire moth ball decays away in a definite time. The time to reach one quarter of the original volume is that in which $V = \frac{1}{4} V_0$ or

$$\frac{4}{3}\pi r^3 = \frac{1}{4} \times \frac{4}{3}\pi r_0{}^3 \ ;$$

namely,

$$r = \frac{r_0}{\sqrt[3]{2}} = r_0 - kt \ ;$$

hence,

$$t = \frac{r_0}{k}\left(1 - \frac{1}{\sqrt[3]{4}}\right) .$$

Similarly the time to reach half-volume, as given in the problem, is

$$10 = \frac{r_0}{k}\left(1 - \frac{1}{\sqrt[3]{2}}\right) ;$$

hence

$$k = \frac{10}{r_0\left(1 - \frac{1}{\sqrt[3]{2}}\right)} .$$

The additional time required for decay to quarter-volume is

$$t - 10 = \frac{10\left(1 - \frac{1}{\sqrt[3]{4}}\right)}{1 - \frac{1}{\sqrt[3]{2}}} - 10$$

$$= 10\,\frac{\frac{1}{\sqrt[3]{2}} - \frac{1}{\sqrt[3]{4}}}{1 - \frac{1}{\sqrt[3]{2}}} = 10\,\frac{1 - \frac{1}{\sqrt[3]{2}}}{\sqrt[3]{2} - 1}$$

$$= \frac{10}{\sqrt[3]{2}} \approx \frac{10}{1.26} \approx 7.94 \text{ days.}$$

17. Consider the differential equation

$$(D_t + a)u = k\,e^{-bt} \tag{1}$$

(compare Equation (5) of Section 9-3(iii)).

(a) Observe for any solution u of (1) that

$$(D_t + b)(D_t + a)u = 0 \tag{2}$$

and show for any solutions v , w of the equations

$$(D_t + b)v = 0 \ , \ (D_t + a)w = 0 \tag{3}$$

that

$$u = v + w \tag{4}$$

is a solution of (2). (Hint: $(D_t + b)(D_t + a) = (D_t + a)(D_t + b)$.)

Since $$D_t(k\,e^{-bt}) = -kbe^{-bt}$$

$$(D_t + b)ke^{-bt} = -kbe^{-bt} + kbe^{-bt} = 0$$

Now, if $(D_t + b)v = 0$ and $(D_t + a)w = 0$ then

$$(D_t + a)[(D_t + b)v] = (D_t + a)0 = 0$$

and $$(D_t + b)[(D_t + a)w] = (D_t + b)0 = 0 \ .$$

Consequently,

$$(D_t + b)(D_t + a)(v + w) = (D_t + b)[(D_t + a)v + (D_t + a)w]$$

$$= (D_t + b)(D_t + a)v = (D_t + a)(D_t + b)v = 0 \ .$$

(b) Use the Form (4) to obtain a solution of (1) satisfying the initial condition

(5) $$u = u_0 \quad \text{at} \quad t = 0 .$$

A solution v of the first equation in (3) has the form $v : t \longrightarrow c_1 e^{-bt}$; a solution w of the second equation, the form $w : t \longrightarrow c_2 e^{-at}$. In the light of Part (a), seek a solution of (1) of the form

$$u = c_1 e^{-bt} + c_2 e^{-at}$$

subject to the initial condition (5). From

$$\begin{aligned}(D_t + a)u &= c_1(D_t + a)e^{-bt} \\ &= c_1(a - b)e^{-bt} = ke^{-bt}\end{aligned}$$

it follows that

$$c_1 = \frac{k}{a - b} .$$

From (5),

$$c_1 + c_2 = u_0 .$$

Consequently, the solution is

$$u = (u_0 - \frac{k}{a - b})e^{-at} + \frac{k}{a - b} e^{-bt} .$$

(c) Show that the solution of (1) satisfying the initial condition (5) is unique.

If u_1 and u_2 are solutions of the differential equation (1) which satisfy the initial condition (5), then $u = u_1 - u_2$ satisfies the differential equation

$$(D_t + a)u = 0$$

and the initial condition

$$u(0) = u_1(0) - u_2(0) = 0 .$$

Consequently, by Theorem 8-5a

$$u(t) = u(0)e^{-at} = 0 .$$

Thus u is 0 ; hence $u_1 = u_2$.

18. The differential equation (1) of the preceding exercise may also be solved as follows. Let v be a solution of the homogeneous equation

$$(D_t + a)v = 0$$

and determine w such that $u = v \cdot w$ is a solution of (1). Following this procedure, determine again the solution of (1) satisfying the initial condition (5) of the preceding exercise.

Take $v : t \longrightarrow e^{-at}$ and set $u = v \cdot w$. Then if u is a solution of (1),

$$(D_t + a)u = v \cdot D_t w ;$$

hence,

$$\frac{dw}{dt} = ke^{(a-b)t}$$

and

$$w(t) = \frac{k}{a-b} e^{(a-b)t} + C .$$

Thus,

$$u(t) = e^{-at} w(t)$$

$$= \frac{k}{a-b} e^{-bt} + C e^{-at} .$$

From the initial condition $u(0) = u_0$, the result of Number 17(b) is obtained.

19. (a) Employ the method of Number 18 to obtain the solution (9) of Equation (1) in Section 9-3(iv) subject to the initial condition $N(0) = N_0$; that is, seek a solution of the form $N(t) = U(t)W(t)$ where $U(t)$ is a solution of the homogeneous equation

$$\frac{dU}{dt} + AU = 0 .$$

(b) Verify that the solution is unique.

A solution of the homogeneous equation is $U : t \longrightarrow e^{-At}$. If $N = U \cdot W$ satisfies the equation

$$\frac{dN(t)}{dt} - A\,N(t) = K\,V(t) ,$$

then

$$U(t)\,\frac{dW(t)}{dt} = e^{-At}\,\frac{dW(t)}{dt} = K\,V(t)$$

and

$$\frac{dW(t)}{dt} = Ke^{At}V(t) ;$$

hence,

$$W(t) = \int_0^t Ke^{Ax} \dot{V}(x)dx + C .$$

Thus, for N ,

$$N(t) = e^{-At}(C + \int_0^t Ke^{Ax}V(x)dx) .$$

From $N(C) = C = N_0$, the result follows.

Solutions Exercises 9-5

1. A reaction in which one molecule each of the reagents A_1 , A_2 , A_3 , ... , A_n combine to form one molecule of the product B is indicated by

$$A_1 + A_2 + \ldots + A_n \xrightarrow{k} B$$

where k is the rate constant in the differential equation (12) of Section 9-4(iv). We write (12) in the form

$$\frac{db}{dt} = ka_1a_2a_3 \ldots a_n$$

where the lower case letters denote the concentrations of the corresponding reagents and the product. If more than one molecule of a given reagent enters into a reaction as for the familiar example

$$2H_2 + O_2 \longrightarrow 2H_2O \qquad (1)$$

then the concentration of a given reagent enters the law of mass section in as many places as the number of its molecules which are involved in building one molecule of the product. Thus, for a reaction in the form of (1)

$$2A_1 + A_2 \longrightarrow B$$

the law of mass action takes the form

$$\frac{db}{dt} = k\, a_1^2\, a_2 . \qquad (2)$$

(a) Write the law of mass action (2) in the form corresponding to (12) of Section 9-4(iv). What are the rates da_1/dt and da_2/dt ? (Hint: Use the fact that the amount of each element is unaltered in a chemical reaction).

Let the concentrations of H_2 , O_2 , and H_2O be a , b , and c , respectively, and let the corresponding initial concentrations be a_0 , b_0 , and 0 . From the constancy of the content of elemental hydrogen and oxygen, the hydrogen balance is given by

$$a + c = a_0 ,$$

and the oxygen balance by

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$$b + \frac{c}{2} = b_0 \cdot\cdot$$

The law of mass action for this reaction is then

$$\frac{dc}{dt} = k\ a^2 b = k(a_0 - c)^2 (b_0 - \frac{c}{2}) .$$

(b) Similarly, for a reaction in which ν_i molecules of the reagent A_i are combined to form one molecule of the product B, we write

$$\nu_1 A_1 + \nu_2 A_2 + \ldots + \nu_n A_n \xrightarrow{k} B$$

and obtain the law of mass action in the form

$$(3) \qquad \frac{db}{dt} = k\ a_1^{\nu_1}\ a_2^{\nu_2} \ldots a_n^{\nu_n} .$$

Write the law of mass action (c) in the form corresponding to (12). Determine da_i/dt in terms of a_i.

Let c_i be the initial concentration of a_i, $(i=1,\ldots,n)$. Since ν_i molecules of A_i enter into each molecule of B,

$$a_i + \nu_i\ b = c_i .$$

Enter this result in (3) to obtain

$$\frac{db}{dt} = k(c_1 - \nu_1 b)^{\nu_1} (c_2 - \nu_2 b)^{\nu_2} \ldots (c_n - \nu_n b)^{\nu_n} .$$

Note also, from $b = \dfrac{c_i - a_i}{\nu_i}$ that it is possible to write the equation in terms of the concentration of any reagent A_i:

$$\frac{da_i}{dt} = -\nu_i \frac{db}{dt} = -\nu_i k[c_1 - \frac{\nu_1}{\nu_i}(c_i - a_i)]^{\nu_1} \ldots [c_n - \frac{\nu_n}{\nu_i}(c_i - a_i)]^{\nu_n} .$$

2. When a reaction product is obtained as the result of a chain of reactions the law of mass action cannot be applied directly to obtain the rate of production. We must take account of the intermediate reactions. The simplest example is given by a reversible unimolecular reaction in which a molecule of A may be converted to a molecule of B with one probability, but a molecule of B may revert to A with another probability. This reaction is indicated by

$$A \underset{k_2}{\overset{k_1}{\rightleftharpoons}} B$$

and the reaction is governed by the equation

(1) $$\frac{db}{dt} = k_1 a - k_2 b$$

where k_1 is the rate constant for conversion of A to B and k_2 is the rate constant for conversion of B to A . Let C_0 denote the initial concentration of B and C_1 the initial concentration of A , and write (1) in the form of (12) of Section 9-4(iv). Describe the course of the reaction. What state is approached as t approaches infinity?

Let a_0 and b_0 be the initial concentrations of a and b , respectively. From conservation of mass,

$$a + b = a_0 + b_0 ;$$

hence (1) can be put in the form

$$\frac{db}{dt} = k_1(a_0 + b_0 - b) - k_2 b$$
$$= k_1(a_0 + b_0) - (k_1 + k_2)b .$$

It follows that

$$b = \frac{1}{k_1 + k_2} [k_1(a_0 + b_0) + (k_2 b_0 - k_1 a_0)e^{-(k_1+k_2)t}] .$$

Consequently,

$$\lim_{t\sim\infty} b = \frac{k_1(a_0 + b_0)}{k_1 + k_2} , \quad \lim_{t\sim\infty} a = \frac{k_2(a_0 + b_0)}{k_1 + k_2} ,$$

and $\frac{b}{a}$ approaches a constant value $\frac{k_1}{k_2}$ which depends only on the rate constants and not on the initial conditions.

3. **A catalyst** is a compound which enters into a reaction, but which is not consumed in the process. The effect of employing a catalyst is to enhance the reaction. Letting E denote the catalyst, the form of a simple catalyzed reaction is

$$E + A \xrightarrow{k} E + B .$$

Since the catalyst is not consumed in the reaction, its concentration $\mathcal{E}$ remains constant and therefore we expect

(1) $$\frac{db}{dt} = k\mathcal{E}a = k\mathcal{E}(C - b)$$

where C is the initial concentration of A .

Life is never so simple; the action of biological catalysts or enzymes does not follow the elementary law (1). It was conjectured by

Michaelis and Menten* that a reagent or substrate A enters into a loosely bound intermediate combination with the enzyme E . The intermediate combination may dissociate into E and A or into E and the reaction product B . Thus an enzymatically catalyzed reaction is assumed to take the form

$$E + A \underset{k_2}{\overset{k_1}{\rightleftharpoons}} I \xrightarrow{k_3} E + B .$$

Verify that a reaction of this form satisfies the algebraic relations

$$\mathcal{E} + I = \mathcal{E}_0 , \quad a + i + b = C , \tag{2}$$

where a , $\mathcal{E}$, i , b denote the concentrations of substrate, enzyme, intermediate, and product, respectively, and at $t = 0$ the initial concentrations are given by

$$\begin{aligned} a(0) &= C , \quad \mathcal{E}(0) = \mathcal{E}_0 , \\ i(0) &= 0 , \quad b(0) = 0 . \end{aligned} \tag{3}$$

Show that the reaction is goverened by the differential equations

$$\begin{aligned} \frac{da}{dt} &= -k_1 \mathcal{E} a + k_2 i \\ \frac{d\mathcal{E}}{dt} &= -k_1 \mathcal{E} a + (k_2 + k_3) i \\ \frac{di}{dt} &= k_1 \mathcal{E} a - (k_2 + k_3) i \\ \frac{db}{dt} &= k_3 i . \end{aligned} \tag{4}$$

The system (3), (4) has not been solved explicitly. In order to obtain an explicit practical solution Briggs and Haldane assumed that the initial amount of substrate C is very large compared to the amount $\mathcal{E}_0$ of available enzyme, and that the concentration i reaches near equilibrium before an appreciable amount of the substrate is converted. The equilibrium assumption requires that, to a good approximation,

$$\frac{di}{dt} = k_1 \mathcal{E} a - (k_2 + k_3) i = 0 . \tag{5}$$

Ignoring the very rapid initial phase of the reaction in which the equilibrium (5) is set up and the terminal phase in which the concentration of substrate is not large compared to $\mathcal{E}_0$, we replace the differential system (4) by

$$\begin{aligned} \frac{da}{dt} &= -k_1 \mathcal{E} a + k_2 i \\ \frac{db}{dt} &= k_3 i , \end{aligned} \tag{6}$$

the algebraic relations (2) by

$$\mathcal{E} + i = \mathcal{E}_0 , \quad a + i + b = C , \quad \mathcal{E} a = k_M i \tag{7}$$

where k_M is the Michaelis constant $k_M = (k_2 + k_3)/k_1$ and the last equation comes from (5). Compatibly with the first of the assumptions above we replace the initial conditions (3) by

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*See Bartholomay, A.F., "Physicomathematical foundations of reaction rate theory" in Physicomathematical Aspects of Biology, Academic Press, N.Y., 1962.

$$\text{(8)} \qquad \begin{aligned} a(0) &= C_1 , \quad \mathcal{E}(0) = C_2 , \\ i(0) &= C_3 , \quad b(0) = 0 . \end{aligned}$$

Here the constants C_1, C_2, C_3 must be compatible. Thus, the constant C_3 corresponds to the relatively small amount of A which is consumed in reaching the equilibrium (5).

The argument we have given in obtaining the modified system (6), (8) is not mathematically complete. To complete the story it is necessary to show how well the solution of the modified system approximates that of the original system. This is a more difficult question which we shall not attempt to answer here. The main point is that the modified system is easy to solve explicitly while the original system, which incorporated all the information at our disposal, is not. In practice the modified system is therefore actually more useful and under the stipulated assumption that a is much greater than $\mathcal{E}_0$ it has proved to be adequate.

Solve the system (6), (8). (Hint: Use the first and third equations in (7) to eliminate $\mathcal{E}$ and then obtain the differential equation for a .) Why is it unnecessary to obtain the differential equation for b in order to solve for b ?

The algebraic relations (2) simply state conservation of mass: the amount of free enzyme plus the amount bound in I is constant; the amount of substrate plus the amount bound in I plus the amount used to make B is constant.

The Equations (4) simply express the sum of the reactions affecting each particular component of the process. Thus the first equation indicates that A is being consumed in the reaction $E + A \xrightarrow{k_1} I$ at the rate $k_1 \mathcal{E} a$ and replenished in the reaction $I \xrightarrow{k_2} E + A$ at the rate $k_2 i$. Note that the differential equations (4) imply the algebraic relations (2).

From the first equation of (7), $\mathcal{E} a = a(\mathcal{E}_0 - i)$, insert this expression in the third equation of (7) to obtain

$$\text{(9)} \qquad i = \frac{a\,\mathcal{E}_0}{k_M + a} .$$

Furthermore, from the first equation of (6) and the third equation of (7)

$$\frac{da}{dt} = (-k_1 k_M + k_2) i = -k_3 i ;$$

whence, from (9),

$$\text{(10)} \qquad \frac{da}{dt} = - \frac{\mathcal{E}_0 k_3 a}{k_M + a} .$$

Similarly, the equation for b may be obtained with the second equation of (7); i.e., $b = C - a - i$. Together with (9) this yields

(11) $$b = C - \frac{k_M i}{\mathcal{E}_0 - i} - i \ .$$

Solve (11) for i in terms of b and insert in the second equation of (6) to obtain the differential equation for b . Equation (11) is quadratic in i and the differential equation will accordingly have a complicated appearance. However, it is necessary to solve for a only by means of (10) and to use the algebraic relations to determine b ; namely,

$$b = C - a - \frac{a\mathcal{E}_0}{k_M + a} \ .$$

For the solution of (10) observe that

$$\frac{k_M + a}{a}\frac{da}{dt} = (\frac{k_M}{a} + 1)\frac{da}{dt} = -\mathcal{E}_0 k_3 \ .$$

Integrate with respect to t :

$$k_M \log a + a + \text{const.} = -\mathcal{E}_0 k_3 \ .$$

Although this equation cannot be solved explicitly for a it is an effective solution since we may easily plot or tabulate the inverse function which gives t in terms of a .

4. A Soviet statistical yearbook reports that the birthrate was 21.2 per 1000 in 1963 and the deathrate 7.2 per 1000 . As of Jan. 1, 1965 the population was officially estimated at 229 million. The official Soviet population projection predicts a population of 250 million in 1970, 263 million in 1975, and 280 million in 1980. Assuming that the projection is based on a constant deathrate, show that the published report predicts an immediate upswing in the country's birthrate.

Let the birthrate be given by $.0212 + b(t)$ where t is measured in years after Jan. 1, 1964. The growth constant is then $[.0212 + b(t)] - .0072 = .014 + b(t)$. Let N_0 be the unspecified population at $t = 0$; then

$$\frac{1}{N}\frac{dN}{dt} = .014 + b(t)$$

and

$$\log \frac{N}{N_0} = .014t + \epsilon(t) \tag{1}$$

where

$$\epsilon(t) = \int_0^t b(t)dt \ .$$

If $b(t) > 0$ over a given time interval, $[t_1,t_2]$ then $\epsilon(t_2) > \epsilon(t_1)$ and if $b(t) < 0$ then $\epsilon(t_2) < \epsilon(t_1)$. Consequently, it may be inferred that forecasters anticipate an increase or decline in the birthrate according to whether $\Delta = \epsilon(t_2) - \epsilon(t_1)$ is positive or negative. Now set $t_2 = t_1 + 5$ in (1) to obtain

$$\log \frac{N_2}{N_0} = .014\ (t_1 + 5) + \epsilon(t_2)$$

$$= \log \frac{N_1}{N_0} + .07 + \epsilon(t_2) - \epsilon(t_1) \ ;$$

whence,

$$\Delta = \epsilon(t_2) - \epsilon(t_1) = \log \frac{N_2}{N_1} - .07 \ .$$

From the accompanying table we obtain for the successive five-year periods,

$Z = N \times 10^{-8}$	log Z
2.29	.8286
2.50	.9163
2.63	.9670
2.80	1.0296

$$1965 - 1970 : \Delta = .0877 - .0700 = .0177 \ ,$$

$$1970 - 1975 : \Delta = .0517 - .0700 = -.0183 \ ,$$

$$1975 - 1980 : \Delta = .0626 - .0700 = -.0074 \ .$$

Thus the forecast is for a rise in birthrate over the 1963 level for the first five-year period and a drop below that level for the next two periods.

5. In this problem it may seem at first that we do not have enough data for its solution, but the situation can be handled in terms of differential equations.

   It began snowing sometime before noon. A snow plow set out to clear a road at noon. It traveled the first mile in one hour and the second mile in two hours. What time did it start snowing? (Assume the snow falls at a constant rate (ft/hr) and that the snow plow removes snow at a constant rate (ft^3/hr), and neglect the compressibility of the snow.)

Let t_0 be the number of hours before noon when the snow fall began. Let x be the position of the plow at t hours after noon. The height of snow at x when the plow reaches it is proportional to $t + t_0$. The rate of snow removal is proportional to the speed of the plow and the height of the snow (the width of the plow is assumed fixed); hence, by the assumption of a constant rate of snow removal

$$(t + t_0)\frac{dx}{dt} = k$$

where k is constant. Integrate

$$\frac{dx}{dt} = \frac{k}{t + t_0}$$

to obtain

$$x = k \log(t + t_0) + C .$$

For the distance x at $t = 0 , 1 , 3$ we have

$$0 = k \log t_0 + C ,$$

$$1 = k \log (t_0 + 1) + C ,$$

$$2 = k \log (t_0 + 3) + C .$$

Use the first of these equations to eliminate C :

$$1 = k \log \frac{t_0 + 1}{t_0} ,$$

$$2 = k \log \frac{t_0 + 3}{t_0} .$$

Now eliminate k :

$$2 \log \frac{t_0 + 1}{t_0} = \log \frac{t_0 + 3}{t_0}$$

or

$$\left(\frac{t_0 + 1}{t_0}\right)^2 = \frac{t_0 + 3}{t_0} ;$$

whence,

$$t_0 = 1 .$$

It began snowing at 11 a.m..

Teacher's Commentary

Chapter 10

INTEGRATION

The first five sections of this chapter are devoted to techniques of formal integration. We try to avoid presenting the subject entirely as a hodgepodge of special techniques but make as full use as possible of whatever generality we can find at this level. Nonetheless, formal integration remains an art and practice is necessary for the basic level of proficiency. The exercises aim at sufficient variety to keep interest alive.

It is possible to over-indulge in the sport of formal integration. We have noted that it is not always possible to obtain a formal integral in terms of elementary functions; often, though it may be possible, it is not worthwhile--especially so when the integrand is simple and the representation of the integral complicated. Often an approximate representation of the integrand and its antiderivative or a numerical computation of a definite integral will serve our purposes better.

The first two parts of Section 10-6 take up techniques for the evaluation of special definite integrals with the aid of symmetries or special end conditions. Section 10-6(iii) extends the concept of integral to include "improper" forms; in this section we have tried to bring the difficulties plainly to the surface so that the student will be unlikely to be trapped in the pitfalls of the subject.

Sections 10-7, 8, 9 treat enough of the elements of differential equations to make the student confident about Chapter 9 and to prepare him for applications to come. It is not traditional to introduce this material so early, but there is little reason for postponement. The enormous flexibility gained in applications of integration is well worth the bit of extra effort here. Furthermore, the subject is a natural outgrowth of the preceding material; its results are easily understood and used; and the theory, to the degree it is developed here, is aimed at the level of those students who have mastered the earlier theoretical passages of the text.

In the solutions of exercises, when the symbol I is introduced without definition it represents the integral of the stated problem.

Solutions Exercises 10-1

1. For each of the following sketch the graph of f defined by the given integral

2. Improve the sketches of Number 1 by employing the following information for corresponding parts (a) - (d).

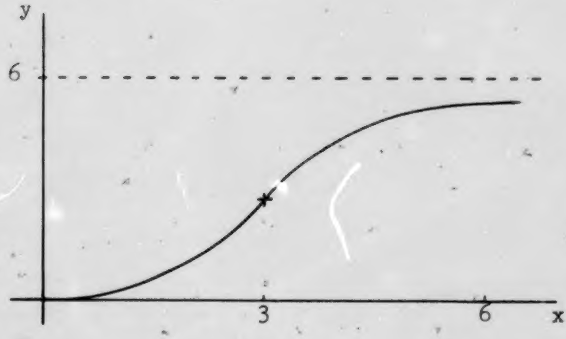
Although each of the integrals (a) - (d) is given in terms of elementary functions, the idea is to make no use of the fact and to avoid point-by-point plotting. The data given in Number 2 were obtained from the explicit integrals and give the values of f at extrema and inflection points.

(a) $f(x) = \int_0^x t^3e^{-t}dt = -(x^3 + 3x^2 + 6x + 6)e^{-x} + 6 , \qquad x \geq 0 .$

Since the integrand is positive f is increasing and has its minimum at $x = 0$.

$$f'(x) = x^3e^{-x} \;;\; f''(x) = (3x^2 - x^3)e^{-x} .$$

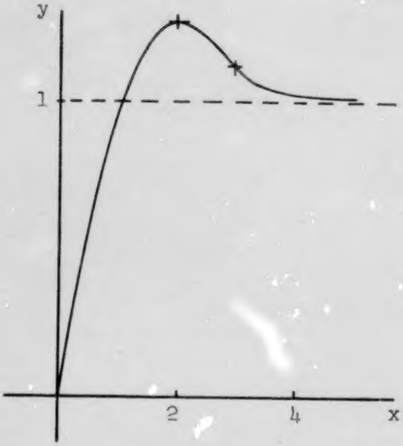
At $x = 3$ the flexure changes from upward to downward. The data from Number 2 locate the inflection point at $(3, 6 - \frac{78}{e^3})$ and indicate the asymptote $y = 6$.



(b) $f(x) = \int_0^x (2 - t)e^{-t}dt = (x - 1)e^{-x} + 1 ,$ $x \geq 0 .$

$$f'(x) = (2 - x)e^{-x} ; f''(x) = (x - 3)e^{-x} .$$

From the data of Number 2 there is a maximum $1 + \frac{1}{e^2}$ at $x = 2$, an inflection point at $(3 , 1 + \frac{2}{e^3})$, and an asymptote, $y = 1$. The change of flexure at $x = 3$ is from downward to upward.

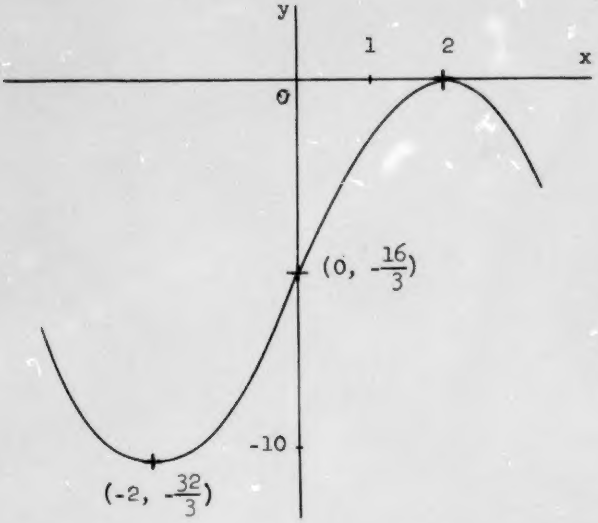


(c) $f(x) = \int_2^x (4 - t^2)dt = -(\frac{x^3}{3} - 4x + \frac{16}{3}) .$

$$f'(x) = 4 - x^2 , f''(x) = -2x .$$

There is a maximum, 0, at $x = 2$; a minimum, $-\frac{32}{3}$, at $x = -2$; a point of inflection $(0,\frac{16}{3})$.

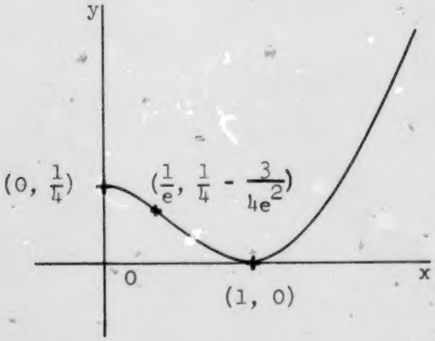
✓



(d) $f(x) = \int_1^x t \log t \, dt = \frac{x^2}{2}(\log x - \frac{1}{2}) + \frac{1}{4}$, $x > 0$.

$$f'(x) = x \log x \; ; \; f''(x) = 1 + \log x \, .$$

The minimum 0 at $x = 1$. The inflection point $(\frac{1}{e}, \frac{1}{4} - \frac{3}{4e^2})$ where the flexure changes from downward to upward.



3. Sketch the graph of $f(x) = \int_0^x \sin \pi t \, dt$.

This exercise can be completed from the explicit representation $f(x) = \frac{1}{\pi}(1 - \cos \pi x)$. It is preliminary to the next exercise.

4. Let $f(x) = \int_0^x \cos^4 t \, dt$. Given that $f(\frac{\pi}{2}) = \frac{3\pi}{16}$ and $f(\pi) = \frac{3\pi}{8}$, sketch the graph of f . (Hint: show first that for all x , $f(x + \pi) = a + f(x)$.)

(Note: $\int_0^x \cos^4 t \, dt = \frac{1}{4} \cos^3 x \sin x + \frac{3}{8} \cos x \sin x + \frac{3}{8} x$.)

$$f(x + \pi) = f(x) + \int_x^{x+\pi} \cos^4 t \, dt .$$

Since the integrand is periodic with period π , the integral in this formula is independent of x (Section 10-6(i)). Consequently,

$$\int_x^{x+\pi} \cos^4 t \, dt = \int_0^\pi \cos^4 t \, dt = f(\pi) = \frac{3\pi}{8} ,$$

as given. To prove the result is independent of x , let n be the integer for which $x \leq n\pi < x + \pi$, $(n = [\![\frac{x}{\pi}]\!])$ and set $x = n\pi + r$. Then

$$\int_x^{x+\pi} \cos^4 t \, dt = \int_x^{n\pi} + \int_{n\pi}^{x+\pi} \cos^4 t \, dt .$$

In the first of these integrals substitute $t = u + n\pi - \pi$; in the second $t = v + n\pi$, to obtain

$$\begin{aligned} \int_x^{x+\pi} \cos^4 t \, dt &= \int_r^\pi \cos^4 u \, du + \int_0^r \cos^4 v \, dv \\ &= \int_0^\pi \cos^4 z \, dz . \end{aligned}$$

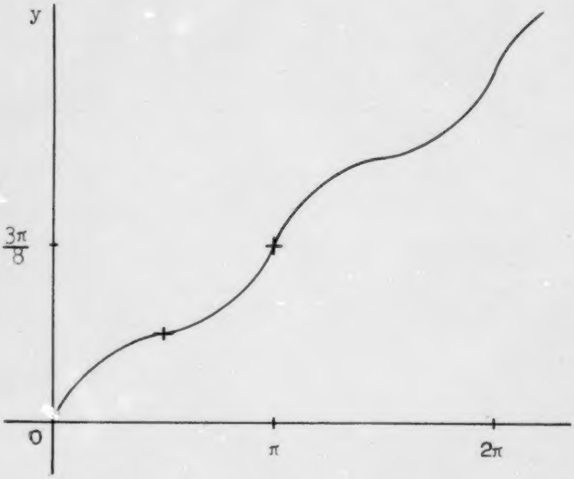
Now observe that

$$\begin{aligned} f(x) = f(r + n\pi) = na + f(r) &= \frac{ax}{\pi} + \{f(r) - \frac{ar}{\pi}\} \\ &= \frac{ax}{\pi} + g(x) \end{aligned}$$

where g is periodic with period π. Thus the entire graph of f is determined by the segment over $[0,\pi]$. Since

$$f'(x) = \cos^4 x \; ; \; f''(x) = -4\cos^3 x \sin x$$

the graph has horizontal tangents at $x = n\pi + \frac{\pi}{2}$, (n integral), but no extrema since f is everywhere increasing. These horizontal tangents occur at inflection points. The remaining inflection points occur at the zeros of $\sin x$, at $x = n\pi$, where the slope is 1.



Solutions Exercises 10-2

1. Integrate in terms of elementary functions when possible.

(a) $\dfrac{x^2}{x^3 + a^3}$. Substitute $u = x^3$.

$$I = \log \sqrt[3]{x^3 + a^3} + C.$$

(b) $x^3 \sqrt[7]{1 - x^4}$. Substitute $u = x^4$.

$$I = \frac{7}{32}(1 - x^4)^{8/7} + C.$$

(c) $\dfrac{x - 1}{\sqrt{x^2 - 2x + a^2}}$. Substitute $u = x^2 - 2x + a^2$.

$$I = \sqrt{x^2 - 2x + a^2} + C .$$

(d) $\dfrac{(a + b\sqrt{x})^{13}}{\sqrt{x}}$. Substitute $u = \sqrt{x}$.

$$I = \frac{(a + b\sqrt{x})^{14}}{7b} + C .$$

The high exponent should force the student to substitute rather than expand the binomial.

(e) $\dfrac{x^{m-1}}{ax^m + b}$. Substitute $u = ax^m + b$.

$$I = \frac{1}{ma} \log |ax^m + b| + C .$$

(f) $\dfrac{1}{x + x^\alpha}$, $\alpha \neq 1$. Compare (e): $\dfrac{1}{x + x^\alpha} = \dfrac{x^{-\alpha}}{x^{1-\alpha} + 1}$.

$$I = \frac{1}{1 - \alpha} \log (x^{1-\alpha} + 1) + C .$$

(g) $\dfrac{x^2 + 1}{x - 1}$. Substitute $u = x - 1$, or write

$$\frac{x^2 + 1}{x - 1} = \frac{(x^2 - 1) + 2}{x - 1} = x + 1 + \frac{2}{x - 1} .$$

$$I = \frac{x^2}{2} + x + 2 \log |x - 1| + C .$$

(h) $\dfrac{(1 + x)^2}{1 + x^2} = \dfrac{(1 + x^2) + 2x}{1 + x^2}$.

$$I = x + \log (1 + x^2) + C .$$

(i) $\dfrac{1}{\sqrt{1 - x^4}}$. An elliptic integral; it cannot be expressed as an elementary function.

(j) $\dfrac{x}{\sqrt{a^4 - x^4}}$. Substitute $u = (\frac{x}{a})^2$.

$$I = \frac{1}{2} \arcsin (\frac{x}{a})^2 + C .$$

(k) $\dfrac{x}{\sqrt{a^4 + x^4}}$. Substitute $u = (\frac{x}{a})^2$.

$$I = \frac{1}{2} \operatorname{argsinh} (\frac{x}{a})^2 + C .$$

(ℓ) $\dfrac{x}{x^4 + a^4}$. Substitute $u = (\frac{x}{a})^2$.

$$I = \frac{1}{2a^2} \arctan (\frac{x}{a})^2 + C .$$

2. Integrate in terms of elementary functions.

(a) $\dfrac{\sin x}{(a + b \cos x)^n}$. Substitute $u = a + b \cos x$.

For $n = 1$, $I = -\frac{1}{b} \log |a + b \cos x| + C$; for $n \neq 1$,

$I = \dfrac{(a + b \cos x)^{1-n}}{b(n - 1)} + C$.

(b) $3 \cos x \sin 2x = 6 \cos^2 x \sin x$.

Set $u = \cos x$. $I = -2 \cos^3 x + C$.

(c) $\sin^\alpha x \cos x$. Substitute $u = \sin x$.

For $\alpha \neq -1$, $I = \dfrac{\sin^{\alpha+1} x}{\alpha + 1} + C$; for $\alpha = -1$, $I = \log |\sin x| + C$.

(d) $\sin^m x \cos^3 x = \sin^m x(1 - \sin^2 x)\cos x$.

Substitute $u = \sin x$.

$$I = \frac{\sin^{m+1} x}{m + 1} - \frac{\sin^{m+3} x}{m + 3} + C .$$

(e) $\sin^2 2x = \frac{1}{2}(1 - \cos 4x)$.

$$I = \frac{x}{2} - \frac{\sin 4x}{8} + C .$$

(f) $\sec^5 ax \tan ax = \dfrac{\sin ax}{\cos^6 ax}$.

$$I = \frac{1}{5a \cos^5 ax} + C .$$

(g) $\frac{\cos \sqrt{2x}}{\sqrt{x}}$. Substitute $u = \sqrt{2x}$.

$$I = \sqrt{2} \sin \sqrt{2x} + C \ .$$

(h) $\sqrt{\frac{\cos x}{\tan x}} = \frac{\cos x}{\sqrt{\sin x}}$, $(0 < x < \frac{\pi}{2})$.

$$I = 2\sqrt{\sin x} + C \ .$$

(i) $\frac{1}{a + b \cos^2 x} = \frac{1}{\cos^2 x} \ \frac{1}{(a + b) + a \tan^2 x}$.

If $a = 0$, $I = \frac{\tan x}{b} + C$. If $a \neq 0$ substitute $u = \tan x$. Then

$$I = \frac{1}{a} \int \frac{du}{u^2 + (1 + \frac{b}{a})}$$

and there are three cases. Put $\lambda = \sqrt{|1 + \frac{b}{a}|}$.

$\frac{b}{a} > -1$, $I = \frac{1}{a} \arctan \frac{\tan x}{\lambda} + C$.

$\frac{b}{a} = -1$, $I = - \frac{1}{a \tan x} + C$.

$\frac{b}{a} < -1$, $I = \frac{1}{2a\lambda} \log \left| \frac{\tan x - \lambda}{\tan x + \lambda} \right| + C$.

3. Integrate in terms of elementary functions.

(a) $2x^3 e^{x^4}$ Substitute $u = x^4$.

$$I = \frac{1}{2} e^{x^4} + C \ .$$

(b) $e^x(a + be^x)^r$, $(b \neq 0)$. Substitute $u = a + be^x$.

For $r \neq -1$, $I = \frac{(a + be^x)^{r+1}}{b(r) + 1} + C$.

For $r = -1$, $I = \frac{1}{b} \log (a + be^x) + C$.

(c) $e^x(a + be^{-x})^3$. Expand the binomial. In contrast to (b). $\int e^x(a + be^{-x})^r \, dx$ is not generally expressible in terms of elementary functions.

$$I = a^3 e^x + 3a^2 bx - 3ab^2 e^{-x} - \frac{b^3}{2} e^{-2x} + C \ .$$

(d) $\dfrac{ae^x}{b + ce^x}$, $c \neq 0$. Compare Part (b) with $r = -1$.

(e) $\dfrac{a}{b + ce^x} = \dfrac{ae^{-x}}{be^{-x} + c}$. Substitute $u = be^{-x} + C$.

$$I = -\frac{a}{b}\log(be^{-x} + c) + K .$$

(f) $\dfrac{\log x}{x}$. Substitute $u = \log x$.

$$I = (\log x)^2 + C .$$

(g) $\dfrac{1}{x \log x}$. Substitute $u = \log x$.

$$I = \log \log x + C .$$

(h) $\dfrac{\log^2 x}{x}$. Substitute $u = \log x$.

$$I = \frac{\log^3 x}{3} + C .$$

(i) $\sinh^m ax \cosh ax$. Set $u = \sinh ax$.

$$I = \frac{\sinh^{m+1} ax}{a(m + 1)} + C .$$

(j) $\cosh^3 ax = (\sinh^2 ax + 1) \cosh ax$.

$$I = \frac{\sinh^3 ax}{3a} + \frac{\sinh ax}{a} + C .$$

(k) $\dfrac{1}{\sinh x + 2 \cosh x} = \dfrac{2e^x}{3e^{2x} + 1}$.

Substitute $u = e^x\sqrt{3}$.

$$I = \frac{2}{\sqrt{3}} \arctan e^x \sqrt{3} + C .$$

(ℓ) $\dfrac{1}{a^2 \sinh^2 x + b^2 \cosh^2 x}$, $b \neq 0$.

Compare Number 2(i). Set $t = \tanh x$, $dt = \dfrac{dx}{\cosh^2 x}$ to get

$$I = \int \frac{dt}{b^2 + a^2t^2} = \begin{cases} \dfrac{1}{ab} \arctan \dfrac{\tanh x}{b} + C , \text{ if } a \neq 0 , \\ \dfrac{1}{b^2} \tanh x + C , \text{ if } a = 0 . \end{cases}$$

(m) $a^x e^{a^x}$. Set $u = a^x$.

$$I = \frac{e^{a^x}}{\log a} + C .$$

4. Integrate in terms of elementary functions.

(a) $\dfrac{x^2}{\sqrt[5]{x + a}}$. Substitute $u = \sqrt[5]{x + a}$;

then $x = u^5 - a$, $dx = 5u^4\, du$.

$$I = 5\int \frac{(u^5 - a)^2 u^4}{u}\, du = 5\int (u^{13} - 2au^8 + a^2u^3) du$$

$$= 5(x + a)^{4/5} \left\{\frac{(x + a)^2}{14} - \frac{2a(x + a)}{9} + \frac{a^2}{4}\right\} + C .$$

(b) $\dfrac{1}{\sqrt[3]{x} - \sqrt[4]{x}}$. Substitute $x = t^{12}$; then

$$I = \int \frac{12t^{11}}{t^4 - t^3}\, dt = 12\int \frac{t^8}{t - 1}\, dt$$

$$= 12\int \left\{\frac{t^8 - 1}{t - 1} + \frac{1}{t - 1}\right\} dt$$

$$= 12 \int (t^7 + t^6 + t^5 + \ldots + 1 + \frac{1}{t - 1}) dt .$$

Consequently,

$$I = 12 \sum_{n=1}^{8} \frac{x^{n/12}}{n} + \log|x^{1/12} - 1| + C .$$

(c) $\dfrac{1}{\sqrt[6]{x} + \sqrt[7]{2x}}$. Substitute $x = t^{42}$ and set $a = \sqrt[7]{2}$.

Then substitute $t = -as$ and proceed as in (b).

$$
\begin{aligned}
I &= \int \frac{42\ t^{35}}{t + a}\, dt = -42 \int \frac{a^{35}\ s^{35}}{s - 1}\, ds \\
&= -42a^{35} \int \left\{ \frac{s^{35} - 1}{s - 1} + \frac{1}{s - 1} \right\} ds \\
&= -42a^{35} \int \left\{ \sum_{n=0}^{34} s^n + \frac{1}{s - 1} \right\} ds \\
&= -42a^{35} \left\{ \sum_{n=1}^{35} \frac{s^n}{n} + \log |s - 1| \right\} + C \\
&= -42 \times 2^5 \left\{ \sum_{n=1}^{35} \frac{(-1)^n}{n} \left(\frac{x}{64}\right)^n + \log\left[1 + \frac{x}{64}\right]^{1/42} \right\} + C\ .
\end{aligned}
$$

(d) $\dfrac{\sin 2x}{1 + 4 \cos^2 x}$. Set $u = \cos^2 x$, $du = -\sin 2x\, dx$.

$$I = -\frac{1}{4} \log(1 + 4 \cos^2 u) + C\ .$$

(e) $x\, e^{x^2} e^{e^{x^2}}$. Set $u = e^{e^{x^2}}$, $du = 2e^{e^{x^2}} e^{x^2}\, x\, dx$.

$$I = \frac{1}{2} e^{e^{x^2}} + C\ .$$

(f) $\dfrac{x + 1}{x(a + xe^x)}$. Set $t = xe^x$, $dt = (x + 1)e^x\, dx$.

$$
\begin{aligned}
I &= \int \frac{(x + 1)e^x\, dx}{(xe^x)(a + xe^x)} = \int \frac{dt}{t(a + t)} \\
&= \frac{1}{a} \int \left(\frac{1}{t} - \frac{1}{t + a}\right) dt \\
&= \frac{1}{a} \log \frac{xe^x}{a + xe^x} + C\ .
\end{aligned}
$$

(g) $\dfrac{(\arctan x)^2}{1 + x^2}$. Set $\theta = \arctan x$, $d\theta = \dfrac{dx}{1 + x^2}$.

$$I = \frac{(\arctan x)^3}{3} + C\ .$$

(h) $\frac{\log(\log x)^2}{x \log x} = \frac{2 \log \log x}{x \log x}$. Set $u = \log \log x$, $du = \frac{1}{x \log x} dx$.

$$I = (\log \log x)^2 + C .$$

(i) $(x + 1)e^x \tan(xe^x)$. Set $\theta = xe^x$.

$$I = \int \frac{\sin \theta}{\cos \theta} d\theta = -\log|\cos xe^x| + C .$$

(j) $\frac{xe^{x^2} \cos[\log(e^{x^2} + 1)]}{e^{x^2} + 1}$. Set $t = \log(e^{x^2} + 1)$.

$$I = \frac{1}{2}\int \cos t \, dt = \frac{1}{2} \sin \log(e^{x^2} + 1) + C .$$

5. Use the idea of Example 10-2f to integrate t^2e^t , t^3e^t . Can you obtain and prove a formula for the indefinite integral of t^ne^t ?

$$D(t^2e^t) = (t^2 + 2t)e^t \text{ , hence}$$

$$\int t^2e^t \, dt = t^2e^t - 2\int te^t \, dt$$

where the latter integral is given by Example 10-2f. Hence,

$$\int t^2e^t \, dt = (t^2 - 2t + 2)e^t + C .$$

Similarly,

$$D(t^3e^t) = (t^3 + 3t^2)e^t ,$$

and, therefore,

$$\int t^3e^t \, dt = t^3e^t - 3\int t^2e^t \, dt$$

$$= (t^3 - 3t^2 + 6t - 6)e^t + C .$$

In general, suppose

$$\int t^n e^t \, dt = (t^n + a_1 t^{n-1} + a_2 t^{n-2} + \dots + a_n)e^t + C .$$

Then differentiate to obtain

$$t^n e^t = \{t^n + (a_1 + n)t^{n-1} + [a_2 + (n-1)a_1]t^{n-2} + \dots + (a_n + a_{n-1})\}e^t .$$

This equation is satisfied if

$$a_1 = -n$$

$$a_2 = -(n-1)a_1 = n(n-1)$$

$$\dots$$

$$a_k = (-1)^k \frac{n!}{(n-k)!}$$

$$\dots$$

$$a_n = (-1)^n n! .$$

Thus,

$$\int t^n e^t dt = e^t \left\{ t^n + n! \sum_{k=1}^{n} \frac{(-1)^k t^{n-k}}{(n-k)!} \right\} + C ,$$

as may be verified by differentiation.

TC10-3. Substitutions of Circular Functions.

An integral of a rational combination of sinh x and cosh x can be transformed into an integral of a rational function by a substitution exploiting the analogy between circular and hyperbolic functions. However, it is simpler to recognize that the integrand is a rational function of e^x (See Exercises 10-3, No. 10).

The integration of $\frac{1}{\cos \theta}$ which occurs in Example 10-3d may be accomplished by the substitution $u = \sin \theta$, as follows:

$$\int \frac{d\theta}{\cos \theta} = \int \frac{\cos \theta}{1 - \sin^2 \theta}\, d\theta \quad \int \frac{du}{1 - u^2}$$

$$= \operatorname{argtanh} \sin \theta + C$$

$$= \frac{1}{2} \log \frac{1 + \sin \theta}{1 - \sin \theta} + C \,.$$

Solutions Exercises 10-3

1. Integrate the following functions, the numbers a and b being positive.

(a) $\frac{\sqrt{a^2 - x^2}}{x^2}$. Set $x = a \cos \theta$.

$$I = -\int \tan^2 \theta \, d\theta = -\int [(1 + \tan^2 \theta) - 1] d\theta$$

$$= \theta - \tan \theta + C$$

$$= \arccos \frac{x}{a} - \frac{\sqrt{a^2 - x^2}}{x} + C \,.$$

(b) $\frac{\sqrt{1 + x^2}}{x^4}$. Set $x = \tan t$.

$$I = \int \frac{\cos t}{\sin^4 t}\, dt = -\frac{1}{3 \sin^3 t} + C$$

$$= -\frac{1}{3}\left(\frac{\sqrt{1 + x^2}}{x}\right)^3 + C \,.$$

(Alternatively, set $u = \frac{1}{x^2}$ to obtain $I = -\frac{1}{2}\int \sqrt{1 + u}\, du$.)

(c) $x^2 \sqrt{a^2 - x^2}$. Set $x = a \sin t$.

$$I = a^4 \int \sin^2 t \cos^2 t \, dt = \frac{a^4}{4} \int \sin^2 2t \, dt$$

$$= \frac{a^4}{8} \int (1 - \cos 4t) dt = \frac{a^4}{8} (t - \frac{1}{4} \sin 4t) + C$$

$$= \frac{a^4}{8} \arcsin \frac{x}{a} + (\frac{1}{4}x^3 - \frac{a^2}{8} x) \sqrt{a^2 - x^2} + C .$$

(d) $\dfrac{1}{x^2 \sqrt{x^2 - a^2}}$. Set $x = a \cosh u$.

$$I = \frac{1}{a^2} \int \frac{du}{\cosh^2 u} = \frac{1}{a^2} \tanh u + C$$

$$= \frac{\sqrt{a^2 - x^2}}{a^2 x} + C .$$

(e) $\dfrac{x}{(x^2 + a^2) \sqrt{x^2 - b^2}}$. Set $x^2 - b^2 = t^2$.

$$I = \int \frac{dt}{a^2 + b^2 + t^2} = \frac{1}{\sqrt{a^2 + b^2}} \arctan \frac{t}{\sqrt{a^2 + b^2}} + C$$

$$= \frac{1}{\sqrt{a^2 + b^2}} \arctan \sqrt{\frac{x^2 - b^2}{a^2 + b^2}} + C .$$

(f) $\dfrac{1}{(x^2 + a^2) \sqrt{a^2 x^2 + 1}}$. Set $x = \frac{1}{a} \sinh t$.

$$I = \int \frac{a \, dt}{a^4 + \sinh^2 t} = a \int \frac{1}{a^4 + (1 - a^4)\tanh^2 t} \frac{1}{\cosh^2 t} dt$$

$$= a \int \frac{ds}{a^4 + (1 - a^4)s^2}$$

where $s = \tanh t$.

If $a = 1$, then $I = s + C = \dfrac{x}{\sqrt{x^2 + 1}} + C$.

If $a = 0$, then $I = - \dfrac{1}{x} + C$.

If $0 < a < 1$, then

$$I = \frac{1}{a\sqrt{1-a^4}} \arctan \frac{\sqrt{1-a^4}}{a^2}\, s + C$$

$$= \frac{1}{a\sqrt{1-a^4}} \arctan \frac{x}{a}\sqrt{\frac{1-a^4}{1+a^2x^2}} + C\ .$$

If $a > 1$, then for $b = \dfrac{a^2}{\sqrt{a^4-1}}$,

$$I = \frac{b^2}{a^3}\int \frac{ds}{b^2-s^2} = \frac{b}{2a^3}\log\left|\frac{b+s}{b-s}\right| + C$$

$$= \frac{b}{2a^3}\log\left|\frac{b\sqrt{1+a^2x^2}+ax}{b\sqrt{1+a^2x^2}-ax}\right|\ .$$

(g) $\dfrac{x+2}{\sqrt{m^2 \pm x^2}} = \dfrac{x}{\sqrt{m^2 \pm x^2}} + \dfrac{2}{\sqrt{m^2 \pm x^2}}$.

Plus sign: $I = \sqrt{m^2+x^2} + 2\ \mathrm{argsinh}\ x + C$.

Minus sign: $I = -\sqrt{m^2-x^2} + 2 \arcsin x + C$.

(h) $x^3\sqrt{(4-x^2)^5}$. Set $\sqrt{4-x^2} = t$.

$$I = \int (t^2-4)t^6\, dt = \frac{-(4-x^2)^{7/2}(8+7x^2)}{63} + C\ .$$

(i) $\dfrac{1}{\sqrt{a^2x-x^2}} = \dfrac{1}{\sqrt{\frac{a^2}{4}-(x-\frac{a^2}{2})^2}}$.

Set $x = \frac{a^2}{2}(\sin\theta + 1)$.

$$I = \arcsin\left(\frac{2x}{a^2} - 1\right) + C\ .$$

(Alternatively, set $x = a^2 \sin^2 \psi$ to obtain $I = 2\psi = 2 \arcsin \frac{\sqrt{x}}{a}$. This is suggested by the preliminary substitution $x = t^2$.)

(j) $$\frac{x^2 + ax + b}{x^2 + 1} = \frac{(x^2 + 1) + \text{[illegible]}\ (b - 1)}{x^2 + \text{[illegible]}} .$$

$$I = x + \frac{a}{2} \log (x^2 + 1) + (b - 1) \arctan x + C .$$

(k) $$\sqrt{a^2x + x^2} = \sqrt{(x + \frac{a^2}{2})^2 - \frac{a^4}{4}} .$$

Set $x + \frac{a^2}{2} = \frac{a^2}{2} \cosh z$.

$$I = \frac{a^4}{4} \int \sinh^2 z\, dz = \frac{a^4}{8} \int (\cosh 2z - 1) dz$$

$$= (x + \frac{a^2}{2}) \sqrt{a^2x + x^2} - \frac{a^4}{8} \operatorname{argcosh}(\frac{2x}{a^2} + 1) + C .$$

2. Let $R(x,y)$ denote a rational function in x and y . Reduce the following integrals to integrals of rational functions.

(a) $\int R(x , \sqrt{ax + b}) dx$, $a \neq 0$. Set $\sqrt{ax + b} = t$.

$$I = \frac{2}{a} \int R(\frac{t^2 - b}{a}, t) t\, dt .$$

(b) $\int R(x , \sqrt[n]{\frac{ax + b}{cx + d}}) dx$, n - an integer, $ad - bc \neq 0$.

Set $\sqrt[n]{\frac{ax + b}{cx + d}} = t$.

$$I = \quad n(ad - bc) \frac{t^{n-1}}{(a - ct^n)^2} R \left\{ \frac{dt^n - b}{a - ct^n} , t \right\} dt .$$

3. Using the result of Number 2, integrate $\frac{x}{\sqrt{ax + b} + (\sqrt{ax + b})^3}$.

$$I = \frac{2}{a^2} \int \frac{(t^2 - b)t}{t + t^3} dt = \frac{2}{a^2} \int (1 - \frac{1 + b}{1 + t^2}) dt$$

$$= \frac{2}{a^2} [\sqrt{ax + b} - (1 + b) \arctan \sqrt{ax + b}] + C .$$

4. Reduce to rational form, $\displaystyle\int \frac{dx}{\sqrt{\frac{1-x}{1+x}} + \sqrt[4]{\frac{1-x}{1+x}}}$.

Use the method of Number 2(b).

$$I = -8\int \frac{t^2\,dt}{(1+t)(1+t^4)^2} .$$

5. Express as elementary functions,

(a) $\displaystyle\int \frac{dx}{\sqrt{x^2+1} + \sqrt{x^2-1}}$.

First note that $\displaystyle\frac{1}{\sqrt{x^2+1} + \sqrt{x^2-1}} = \frac{\sqrt{x^2+1} - \sqrt{x^2-1}}{2}$.

$$I = \frac{x}{4}\left(\sqrt{x^2+1} - \sqrt{x^2-1}\right) + \frac{1}{4}\log\left|(x+\sqrt{x^2+1})(x+\sqrt{x^2-1})\right| + C .$$

(b) $\displaystyle\int \frac{dx}{1+\sin x} = \int \frac{1-\sin x}{\cos^2 x}\,dx = \tan x - \frac{1}{\cos x} + C$.

(c) $\displaystyle\int \frac{dx}{1-\cos 2x} = \int \frac{1+\cos 2x}{\sin^2 2x}\,dx = -\frac{1}{2}\left(\cot 2x + \frac{1}{\sin 2x}\right) + C$.

(d) $\displaystyle\int \frac{dx}{x\sqrt[4]{1+x^4}}$. Set $t = \sqrt[4]{1+x^4}$; then

$$\begin{aligned}
I &= \int \frac{t^2}{t^4-1}\,dt = \frac{1}{2}\int \frac{(t^2+1)+(t^2-1)}{t^4-1}\,dt \\
&= \frac{1}{2}\int \left[\frac{1}{t^2-1} + \frac{1}{t^2+1}\right]dt \\
&= \frac{1}{4}\log\left|\frac{t-1}{t+1}\right| + \frac{1}{2}\arctan t + C \\
&= \frac{1}{4}\log\frac{\sqrt[4]{1+x^4}-1}{\sqrt[4]{1+x^4}+1} + \frac{1}{2}\arctan\sqrt[4]{1+x^4} + C .
\end{aligned}$$

(e) $\int \frac{dx}{\sqrt[4]{1 + x^4}}$.   Set $x = \frac{1}{u}$; then

$$I = -\int \frac{du}{u \sqrt[4]{1 + u^4}}$$

and the problem is reduced to (d).

6. (a) The integral $\int \frac{P(x)}{\sqrt{ax^2 + 2bx + c}}\, dx$ where $P(x)$ is a polynomial of degree n and $a \neq 0$ can be reduced to a rational trigonometric form as described in the text. It can also be reduced to the integration of $\frac{1}{\sqrt{ax^2 + 2bx + c}}$; namely, for some polynomial Q of degree $n - 1$ and constant k ,

$$\frac{P(x)}{\sqrt{ax^2 + 2bx + c}} = D_x\{Q(x)\sqrt{ax^2 + 2bx + c}\} + \frac{k}{\sqrt{ax^2 + 2bx + c}} .$$

Show how to find Q and k .

Since

$$D_x(Q(x)\sqrt{ax^2 + 2bx + c}) = \frac{Q'(x)(ax^2 + 2bx + c) + Q(x)(ax + b)}{\sqrt{ax^2 + 2bx + c}} ,$$

the polynomial Q and the constant k must sastisfy

(1) $P(x) \equiv Q(x)(ax + b) + Q'(x)(ax^2 + 2bx + c) + k$.

The constant k and the coefficients of $Q(x)$ can be found from the coefficients of $P(x)$ as follows. Set $P(x) = \sum_{\nu=0}^{n} p_\nu x^{n-\nu}$ and $Q(x) = \sum_{\nu=0}^{n-1} q_\nu x^{n-1-\nu}$. Equate coefficients of like powers on the right and left in (1) beginning with the coefficients of x^n :

$$p_0 = aq_0 + a(n - 1)q_0$$

to obtain

$$q_0 = \frac{p_0}{an} .$$

Thus the leading coefficient of $Q(x)$ is determined. Next, $p_1 = (aq_1 + bq_0) + [a(n - 2)q_1 + 2b(n - 1)q_0]$, or $a(n - 1)q_1 = p_1 - b(2n - 1)q_0$. Similarly all succeeding coefficients are determined step-by-step from the preceding ones:

$$a(n - \nu)q_\nu = p_\nu - b(2n - 2\nu - 1)q_{\nu-1} - c(n - \nu + 1)q_{\nu-2} ,$$

for $\nu = 2 , \dots , n - 1$. Finally, for the constant k ,

$$k = p_0 - bq_0 - cq_1 .$$

(b) Using (a), integrate $\dfrac{t^5 - t^3 + t}{\sqrt{1 - t^2}}$.

Set $Q(t) = q_0t^4 + q_1t^3 + q_2t^2 + q_3t + q_4$ and $P(t) = t^5 - t^3 + t$.

$$D(Q(t)\sqrt{1 - t^2}) = \frac{1}{\sqrt{1 - t^2}}[-5q_0t^5 - 4q_1t^4 + (4q_0 - 3q_2)t^3$$
$$+ (3q_1 - 2q_3)t^2 + (2q_2 - q_4)t + q_3] .$$

Now solve for the coefficients in succession to obtain

$$q_0 = -\frac{1}{5} , q_1 = 0 , q_2 = \frac{1}{15} , q_3 = 0 ,$$
$$q_4 = -\frac{13}{15} , k = 0 .$$

Consequently,

$$I = \int \frac{t^5 - t^3 + t}{\sqrt{1 - t^2}} dt = -\frac{1}{15}(3t^4 - t^2 + 13)\sqrt{1 - t^2} + C .$$

(c) Find the integral of (b) by using trigonometric substitutions, and compare the merits of the two methods.

Set $t = \sin\theta$. Then

$$I = \int (\sin^4\theta - \sin^2\theta + 1) \sin\theta \, d\theta$$

$$= \int [(1 - \cos^2\theta)^2 - (1 - \cos^2\theta) + 1] \sin\theta \, d\theta$$

$$= \int [(\cos^4\theta - \cos^2\theta + 1] \sin\theta \, d\theta$$

$$= -\frac{1}{5}\cos^5\theta + \frac{1}{3}\cos^3\theta - \cos\theta + C$$

$$= -\frac{1}{15}(3t^4 - t^3 + 13)\sqrt{1 - t^2} + C .$$

which is the result obtained in (b). If even powers appeared in $P(t)$ the work in (c) would be more complicated while the work in (b) would not change greatly. Furthermore the method of (a) eliminates the repetitive use of the binomial theorem when P is of high degree.

7. It is stated at the beginning of this section that an integral of any rational combination of x and $\sqrt{Q(x)}$ where $Q(x) = Ax^2 + Bx + C$ can be transformed into a rational combination of circular functions. Yet Theorem 10-3a takes up the case $A \neq 0$ only. Prove the result is true if $A = 0$.

   If $B \neq 0$, set $x = u^2$ to reduce the integral to the case of Theorem 10-3a. Alternatively, first transform to an integral of a rational function by the method of Number 2(a); i.e., set $x = \frac{(t^2 - C)}{B}$; then set $t = f(u)$ where f is any circular function.

   If $B = 0$, then $\sqrt{Q(x)}$ is constant and we already have the integral of a rational function. In this case set $x = f(u)$ where f is any circular function.

8. The proof of Theorem 10-3a does not treat the cases

   (i) $A < 0$, $\frac{C}{A} - \frac{B^2}{4A^2} > 0$,

   (ii) $A = 0$,

   (iii) $\frac{C}{A} - \frac{B^2}{4A^2} = 0$.

   Why not?

   In Case (i), $\sqrt{Q(x)}$ is imaginary for all x.
   Case (ii) can be handled more simply by the method of Number 2(a); also see Number 7.
   In Case (iii), if $A > 0$, then $Q(x)$ is in the form of a square: $Q(x) = (\alpha x + \beta)^2$ and $\sqrt{Q(x)} = |\alpha x + \beta|$ so that the integrand is rational; if $A < 0$, $Q(x) = -(\alpha x + \beta)^2$ so that $\sqrt{Q(x)}$ is real for only one number x.

9. State and verify the result corresponding to Theorem 10-3a for hyperbolic substitutions.

An integral of any rational combination of x and $\sqrt{Q(x)}$ where $Q(x) = Ax^2 + Bx + C$ can be transformed by a substitution $x = f(t)$, where f is a hyperbolic function, into an integral of a rational combination of $\sinh t$ and $\cosh t$.

For the proof consider the same linear substitution $x = u - b$ to obtain the same standard forms as in Theorem 10-3a.

Case (i): $$I = \int \phi(u - b, c\sqrt{a^2 - u^2})du .$$

Set $u = a \tanh t$ to obtain

$$I = a\int \frac{\phi(a \tanh t - b, \frac{ac}{\cosh^2 t})}{\cosh^2 t} dt .$$

Case (ii): $$I = \int \phi(u - b, c\sqrt{u^2 - a^2})du .$$

Set $u = a \cosh t$ to obtain

$$I = a\int \phi(a \cosh t - b, ac \sinh t)\sinh t \, dt .$$

Case (iii): $$I = \int \phi(u - b, c\sqrt{a^2 + u^2})du .$$

Set $u = a \sinh t$ to obtain

$$I = a\int \phi(a \sinh t - b, ac \cosh t)\cosh t \, dt .$$

10. Using a hyperbolic substitution similar to (9) show how to transform the integral of any rational combination of $\sinh x$ and $\cosh x$ into an integral of a rational function.

Consider
$$I = \int \psi(\sinh x, \cosh x)dx .$$
Set $u = \tanh \frac{x}{2}$. Then

$$\sinh x = \frac{2u}{1 - u^2}, \quad \cosh x = \frac{1 + u^2}{1 - u^2}$$

$$dx = \frac{2}{1 - u^2} du .$$

Consequently,

$$I = \int \psi\left(\frac{2u}{1 - u^2}, \frac{1 + u^2}{1 - u^2}\right) \frac{2}{1 - u^2}\, du .$$

Alternatively, note that

$$\psi(\sinh x, \cosh x) = \psi\left(\frac{e^x - e^{-x}}{2}, \frac{e^x + e^{-x}}{2}\right)$$

$$= R(e^x)$$

where R is a rational function. Set $x = \log u$ to obtain

$$I = \int \frac{R(u)}{u}\, du .$$

11. Integrate

(a) $\frac{1}{\sin x}$.

Use $x = 2 \arctan t$ to obtain

$$I = \int \frac{dt}{t} = \log\left|\tan \frac{x}{2}\right| + C .$$

(b) $\frac{1}{\cos x}$ (by a method other than that of Example 10-3d).

Use $x = 2 \arctan t$ to obtain

$$I = \int \frac{dt}{1 - t^2} = \frac{1}{2}\int \left[\frac{1}{1 - t} + \frac{1}{1 + t}\right] dt$$

$$= \frac{1}{2} \log \left|\frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}}\right| + C .$$

Alternatively, from $\cos x = \sin u$, where $u = x + \frac{\pi}{2}$, obtain from the solution of Part (a),

$$I = \int \frac{du}{\sin u} = \log\left|\tan \frac{1}{2}\left(x + \frac{\pi}{2}\right)\right| + C .$$

TC10-4. Integration by Parts.

THEOREM 10-4. When we say "ϕ and ψ are inverses" we mean ϕ is the inverse of ψ and ψ is the inverse of ϕ. (See, also, Exercises 10-4, No. 2.)

Example 10-4b. It might be appropriate to first consider $\int \arcsin x \, dx$. We observe that

$$\arcsin x = (\arcsin x)(1) = (\arcsin x)\frac{dx}{dx}.$$

Thus $\arcsin x = u\frac{dv}{dx}$ where $u = \arcsin x$ and $v = x$. Integrating by parts we get

$$\int \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1 - x^2}} dx$$

$$= x \arcsin x + \sqrt{1 - x^2} + C.$$

It should be noted that the discussion in the text establishes the integrability of $x^n \arcsin x$.

Solutions Exercises 10-4

1. Integrate the following functions.

In these solutions the integral of the problem is written $I = \int u \, dv$.

(a) $x \sin 3x$. $\quad I = -x\frac{\cos 3x}{3} + \frac{\sin 3x}{9} + C.$

(b) $x \cdot 5^x$. $\quad I = \frac{x \cdot 5^x}{\log 5} - \frac{5^x}{(\log 5)^2} + C.$

(c) x^3e^{-2x}. Set $dv = e^{-2x}dx$ and integrate by parts three times.

$$I = -e^{-2x}\left(\frac{x^3}{2} + \frac{3x^2}{4} + \frac{3x}{4} + \frac{3}{8}\right) + C.$$

(d) $\sqrt{x} \log ax$. Set $u = \log ax$, $dv = \sqrt{x}\, dx$.

$$I = \frac{2}{3}x^{3/2}\left(\log ax - \frac{2}{3}\right) + C.$$

Alternatively, set $t = ax$ and use the result of Example 10-4c.

(e) $\log^2 bx$.   Set $u = \log^2 bx$.

$$I = x \log^2 bx - 2\int \log bx \, dx + C$$
$$= x \log^2 bx - 2x(\log bx - 1) + C$$

where the integration of Example 10-4a is used at the end.

(f) $\log^3 x$.   Set $u = \log^3 x$. Apply (e).

$$I = x(\log^3 x - 3 \log^2 x + 6 \log x - 6) + C \, .$$

(g) $\arccos 7x$.   Set $u = \arccos 7x$. Then

$$I = x \arccos 7x - \frac{\sqrt{1 - 49x^2}}{7} + C \, .$$

(h) argsinh ax .   Set $u = \operatorname{argsinh} ax$.

$$I = x \operatorname{argsinh} ax - \frac{1}{a}\sqrt{1 + a^2x^2} + C \, .$$

(i) argtanh bx .   Set $u = \operatorname{argtanh} bx$.

$$I = x \operatorname{argtanh} bx + \frac{1}{b} \log(1 - b^2x^2) + C \, .$$

(j) argtanh $\sqrt{bx}$.   Set $u = \operatorname{argtanh} \sqrt{bx}$.

$$I = x \operatorname{argtanh} \sqrt{bx} - \frac{1}{2}\int \frac{\sqrt{bx}}{1 - bx} \, dx$$
$$= x \operatorname{argtanh} \sqrt{bx} - \frac{1}{b}\int \frac{t^2}{1 - t^2} \, dt$$

where $t = \sqrt{bx}$. From $\dfrac{t^2}{1 - t^2} = \dfrac{1}{1 - t^2} - 1$, obtain

$$I = \left(x - \frac{1}{b}\right) \operatorname{argtanh} \sqrt{bx} + \sqrt{\frac{x}{b}} + C \, .$$

(k) $\arctan \sqrt[3]{x}$. Set $u = \arctan \sqrt[3]{x}$.

$$I = x \arctan \sqrt[3]{x} - \frac{1}{3}\int \frac{x^{1/3}}{1 + x^{2/3}}\, dx .$$

Set $x = z^3$, then

$$\frac{1}{3}\int \frac{x^{1/3}}{1 + x^{2/3}}\, dx = \int \frac{z^3}{1 + z^2}\, dz$$

$$= \frac{1}{2}\int \frac{t}{1 + t}\, dt \qquad (\text{where } t = z^2)$$

$$= \frac{1}{2} \quad (1 - \frac{1}{1 + t})dt$$

$$= \frac{1}{2}(t - \log|1 + t|) .$$

Consequently,

$$I = x \arctan \sqrt[3]{x} - \frac{x^{2/3}}{2} - \log(x^{2/3} + 1) + C .$$

(*l*) $x \arctan x$. Set $u = \arctan x$, $dv = x\, dx$.

$$I = \frac{x^2}{2} \arctan x - \frac{1}{2}\int \frac{x^2}{1 + x^2}\, dx .$$

From $\dfrac{x^2}{1 + x^2} = 1 - \dfrac{1}{1 + x^2}$, obtain

$$I = \frac{1}{2}[(x^2 + 1)\arctan x - x] + C .$$

(m) $\dfrac{\arccos \frac{x}{m}}{\sqrt{x + m}}$. Set $u = \arccos \frac{x}{m}$, $v = 2\sqrt{x + m}$.

$$I = 2\sqrt{x + m} \arccos \frac{x}{m} + 2\int \frac{\sqrt{x + m}}{\sqrt{m^2 - x^2}}\, dx .$$

$$= 2\sqrt{x + m} \arccos \frac{x}{m} + 2\int \frac{dx}{\sqrt{m - x}}$$

$$= 2\sqrt{x + m} \arccos \frac{x}{m} - 4\sqrt{m - x} + C .$$

(n) $x \sin^2 x = \frac{1}{2}(1 - \cos 2x)$.

$$I = \frac{x^2}{4} - \frac{1}{2}\int x \cos 2x \, dx \ .$$

Set $u = x$, $dv = \cos 2x \, dx$, then

$$I = \frac{x^2}{4} - \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + C \ .$$

(o) $x^2 \sin x$. Set $u = x^2$, $v = -\cos x$.

$$I = -x^2 \cos x + \int 2x \cos x \, dx \ .$$

Now set $u = x$, $v = \sin x$.

$$\int 2x \cos x \, dx = 2x \sin x - 2\int \sin x \, dx \ ;$$

whence,

$$I = -(x^2 + 2)\cos x + 2x \sin x + C \ .$$

(p) $x^2 \arcsin ax$. Set $u = \arcsin ax$, $v = \frac{x^3}{3}$.

$$I = \frac{x^3}{3} \arcsin ax - \frac{a}{3}\int \frac{x^3}{\sqrt{1 - a^2x^2}} \, dx \ .$$

Now set $z = \sqrt{1 - a^2x^2}$, $x^2 = \frac{1 - z^2}{a^2}$, $xdx = -\frac{z \, dz}{a^2}$. Then

$$\int \frac{x^3}{\sqrt{1 - a^2x^2}} \, dx = -\frac{1}{a^4}\int (1 - z^2)dz \ ;$$

whence

$$I = \frac{x^3}{3} \arcsin ax - \frac{\sqrt{1 - a^2x^2}}{3a^3} + \frac{(\sqrt{1 - a^2x^2})^3}{9a^3} + C \ .$$

(q) $\cos^3 2x$. Follow the method of Example 10-4g.

More simply, note that

$$\cos^3 2x = (1 - \sin^2 2x)\cos 2x$$

$$I = \frac{\sin 2x}{2} - \frac{\sin^3 2x}{6} + C .$$

(r) $\sin^5 x$. Either follow the method of Example 10-4g or use

$$\sin^5 x = (1 - \cos^2 x)^2 \sin x .$$

$$I = -\frac{\cos^5 x}{5} + \frac{2\cos^3 x}{3} - \cos x + C .$$

(s) $\sin(\log ax)$. Set $u = \sin(\log ax)$.

$$I = x \sin(\log ax) - \int a \cos(\log ax)dx .$$

Similarly,

$$\int \cos(\log ax)dx = x \cos(\log ax) + aI ;$$

whence,

$$I = x \sin(\log ax) - ax \cos(\log ax) - a^2 I$$

and

$$I = \frac{x}{1 + a^2} [\sin(\log ax) - a \cos(\log ax)] + C .$$

(t) $x \tan^2 x$. Set $u = x$, $v = \tan x - x$.

$$I = x \tan x - x^2 - \int (\tan x - x)dx$$

$$= x \tan x + \log |\cos x| - \frac{x^2}{2} + C .$$

(u) $(\arcsin x)^2$. Substitute $x = \sin t$ to obtain $I = \int t^2 \cos t\, dt$ and integrate by parts twice. Alternatively, set $u = (\arcsin x)^2$ to obtain

$$I = x(\arcsin x)^2 - 2\int \frac{x}{\sqrt{1 - x^2}} \arcsin x\, dx .$$

Repeat, to obtain

$$\int \frac{x}{\sqrt{1 - x^2}} \arcsin x\, dx = -\sqrt{1 - x^2} \arcsin x + \int dx .$$

$$I = x(\arcsin x)^2 + 2\sqrt{1 - x^2} \arcsin x - 2x + C .$$

(v) $\sin ax \cos bx$, $(a^2 \neq b^2)$. Set $u = \sin ax$, $v = \frac{\sin bx}{b}$.

$$I = \frac{\sin ax \sin bx}{b} - \frac{a}{b}\int \cos ax \sin bx\, dx .$$

Now set $u = \cos ax$, $v = -\frac{\cos bx}{b}$.

$$\int \cos ax \sin bx\, dx = -\frac{\cos ax \cos bx}{b} - \frac{a}{b} I .$$

Consequently,

$$I = \frac{\sin ax \sin bx}{b} + \frac{a}{b^2} \cos ax \cos bx + \frac{a^2}{b^2} I ;$$

whence

$$I = \frac{1}{b^2 - a^2} [a \cos ax \cos bx + b \sin ax \sin bx] .$$

More simply, note that

$$\sin ax \cos bx = \frac{1}{2}[\sin(a + b)x + \sin(a - b)x] .$$

Hence

$$I = -\frac{1}{2}\left[\frac{\cos(a + b)x}{a + b} + \frac{\cos(a - b)x}{a - b}\right] + C .$$

2. Support the geometrical interpretation of integration by parts by showing for $u = f(x)$ and $v = g(x)$ where f and g have inverses, that $u = \phi(v)$ and $v = \psi(u)$ where ϕ and ψ are inverse functions.

Let F be the inverse of f, G the inverse of g. The functions

$$\phi = fG : v \to fG(v) = f(x) = u$$
$$\psi = gF : u \to gF(u) = g(x) = v$$

are inverses.

3. Verify as alleged after Example 10-4b that the method of the example does demonstrate the reducibility of $\int x^n f(x)dx$ to the integral of a rational function if f is any inverse circular or hyperbolic function, or if f is the logarithmic function.

Set $u = f(x)$, $v = \frac{x^{n+1}}{n+1}$. Let F be the inverse of F. Then

$$v = \frac{F(u)^{n+1}}{n+1} .$$

If F is a trigonometric or hyperbolic function, then $\int v\,du$ is reducible to the integral of a rational function by Theorem 10-3b or Exercises 10-3, Number 10.

If $u = \log x$, then $F(u) = e^u$. Here,

$$\int v\,du = \int \frac{e^{(n+1)u}}{n+1}\,du = \frac{e^{(n+1)u}}{(n+1)^2} + C$$
$$= \frac{x^{n+1}}{(n+1)^2} + C .$$

Thus, explicitly,

$$\int x^n \log x\,dx = \frac{x^{n+1}}{n+1}\left(\log x - \frac{1}{n+1}\right) + C .$$

4. Establish recurrence relations for each of the following (in each case m and n are positive integers).

(a) $\int \sin^n x \, dx$.

Proceed as in Example 10-4g. Otherwise use the result of the example:

$$I_n = \int \cos^n(x - \tfrac{\pi}{2})dx$$

$$= - \frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2} .$$

(b) $\int x^m \log^n x \, dx$. Set $u = \log^n x$, $v = \frac{x^{m+1}}{m+1}$.

$$I_{n,m} = \frac{x^{m+1}}{m+1} \log^n x - \frac{n}{m+1} I_{n-1,m} .$$

$$I_{0,m} = \frac{x^{m+1}}{m+1} .$$

(c) $\int \sin^m x \cos^n x \, dx$. Set $u = \cos^{n-1} x$, $v = \frac{\sin^{m+1} x}{m+1}$.

$$I_{m,n} = \frac{\cos^{n-1}x \sin^{m+1}x}{m+1} + \frac{n-1}{m+1} \int \sin^{m+2}x \cos^{n-2}x \, dx .$$

But, since $\sin^{m+2}x = (1 - \cos^2 x)\sin^m x$,

$$I_{m,n} = \frac{\cos^{n-1}x \sin^{m+1}x}{m+1} + \frac{n-1}{m+1} (I_{m,n-2} - I_{m,n}) ;$$

whence,

$$I_{m,n} = \frac{\cos^{n-1}x \sin^{m+1}x}{m+n} + \frac{n-1}{m+n} I_{m,n-2} .$$

Note that $I_{m,0}$ is given by (a).

It is also possible to reduce first m then n . Instead of proceeding by the given method, use

$$\sin^m x \cos^n x = (-1)^m \cos^m (x - \tfrac{\pi}{2}) \sin^m (x - \tfrac{\pi}{2})$$

to obtain by the preceding result

$$I_{n,m} = - \frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n} .$$

(d) $\int x^n \arctan x \, dx$. Set $u = \arctan x$, $v = \dfrac{x^{n+1}}{n+1}$. Then

(1) $$I_n = \frac{x^{n+1}}{n+1} \arctan x - \frac{1}{n+1} \int \frac{x^{n+1}}{1+x^2} \, dx .$$

But

(2) $$\frac{x^{n+1}}{1+x^2} = \frac{(1+x^2)x^{n-1} - x^{n-1}}{1+x^2} .$$

Insert this in (1) to obtain

$$I_n = \frac{x^{n+1}}{n+1} \arctan x - \frac{x^n}{n(n+1)} + \frac{1}{n+1} \int \frac{x^{n-1}}{1+x^2} \, dx$$

where the integral is given by (1) in terms of I_{n-2} . From this,

$$I_n = \frac{x^{n-1}}{n+1} [(1+x^2) \arctan x - \frac{x}{n}] - \frac{n-1}{n+1} I_{n-2} .$$

Alternatively, reduce the integral in (1) by (2)

$$J_{n+1} = \int \frac{x^{n+1}}{1+x^2} \, dx = \frac{x^n}{n} - J_{n-1}$$

$$= \frac{x^n}{n} - \frac{x^{n-2}}{n-2} + \frac{x^{n-4}}{n-4} - \dots .$$

If n is odd then the sum terminates with $(-1)^{\frac{n-1}{2}} J_1$ where $J_1 = \int \dfrac{x}{1+x^2} \, dx = \frac{1}{2} \log(1+x^2)$. If n is even then the sum terminates with $(-1)^{n/2} J_0$ where $J_0 = \int \dfrac{dx}{1+x^2} = \arctan x$.

(e) $\int x^n \operatorname{argsinh} x \, dx$. First substitute $x = \sinh t$ to obtain

$$I_n = \int t \sinh^n t \cosh t \, dt .$$

Now set $u = t$, $v = \dfrac{\sinh^{n+1} t}{n+1}$; then

(1) $$I_n = t \frac{\sinh^{n+1} t}{n+1} - \frac{1}{n+1} \int \sinh^{n+1} t \, dt .$$

Then proceed as in Example 10-4g to obtain

$$J_{n+1} = \int \frac{\sinh^{n+1} t}{n + 1}\, dt$$

$$= \frac{\sinh^n t \cosh t}{(n + 1)^2} - \frac{n(n - 1)}{(n + 1)^2} J_{n-1} .$$

But, from (1)

$$J_{n-1} = \frac{t \sinh^{n-1} t}{n - 1} - I_{n-2} .$$

Combine these results to obtain

$$I_n = \frac{t \sinh^{n+1} t}{n + 1} - \frac{\sinh^n t \cosh t}{(n + 1)^2} + \frac{n}{(n+1)^2} t \sinh^{n-1} t - \frac{n}{(n+1)^2} I_{n-2}$$

$$= \frac{x^{n-1}}{n+1}(x^2 + \frac{n}{n+1}) \operatorname{argsinh} x - \frac{x^n\sqrt{1 + x^2}}{(n + 1)^2} - \frac{n}{(n - 1)^2} I_{n-2} .$$

(f) $\int x^n \operatorname{argtanh} x\, dx$. Proceed as in (d).

$$I = \frac{x^{n+1}}{n + 1} \operatorname{argtanh} x - \frac{1}{n + 1} J_{n+1} ,$$

where

$$J_{n+1} = \int \frac{x^{n+1}}{1 - x^2}\, dx = \int \frac{x^{n-1} - (1 - x^2)x^{n-1}}{1 - x^2}$$

$$= J_{n-1} - \frac{x^n}{n} .$$

(g) $\int x^n e^{ax}\, dx$. Set $u = x^n$, $v = \frac{e^{ax}}{a}$.

$$I_n = \frac{x^n e^{ax}}{a} - \frac{n - 1}{a} I_{n-1} .$$

(h) $\int x^n \arcsin x\, dx$. Proceed as in (e).

$$I_n = \frac{x^{n-1}}{n + 1} (x^2 - \frac{n}{n + 1})\arcsin x + \frac{x^n\sqrt{1 - x^2}}{(n + 1)^2} + \frac{n(n - 1)}{(n + 1)^2} I_{n-2} .$$

(i) $\int \frac{1}{\sin^n x} dx$. Set $u = \frac{1}{\sin^{n-2} x}$, $v = - \frac{\cos x}{\sin x}$,

Then $$I_n = \frac{\cos x}{\sin^{n-1} x} - (n - 2) \int \frac{\cos^2 x}{\sin^n x} dx .$$

Set $\cos^2 x = 1 - \sin^2 x$ in the last integral to obtain

$$I_n = - \frac{1}{n - 1} \frac{\cos x}{\sin^{n-1} x} + \frac{n - 2}{n - 1} I_{n-2} .$$

(j) $\int \frac{e^x}{x^n} dx$. Set $u = e^x$, $dv = \frac{1}{x^n} dx$.

$$I_n = - \frac{1}{n - 1} \frac{e^x}{x^n} + \frac{a}{n - 1} I_{n-1} .$$

(k) $\int x^n \cos x \, dx$. Set $u = x^n$, $v = \sin x$.

$$I_n = x^n \sin x - n \int x^{n-1} \sin x \, dx .$$

Now set $u = x^{n-1}$, $v = - \cos x$.

$$\int x^{n-1} \sin x \, dx = -x^{n-1} \cos x + (n - 1) \int x^{n-2} \cos x \, dx .$$

Consequently,

$$I_n = x^n \sin x + n x^{n-1} \cos x - n(n - 1) I_{n-2} .$$

For n even, the expansion of I_n terminates with $I_0 = \sin x + C$; for n odd, the expansion ends with $I_1 = x \sin x + \cos x + C$.

TC10-5. Integration of Rational Functions.

The integral of a rational function is an elementary function, but the computation of such an integral may involve considerable labor depending on the complexity of the function at hand.

Every polynomial with real coefficients has a unique factorization of the form given by Equation (4), but to obtain this form one must first find the zeros of Q. The polynomials which appear in exercises in various textbooks are manufactured artifically for purposes of illustration: they are either given in a factored form, or have zeros which can be found easily. In problems arising in applications this is often not the case.

The method of equated coefficients (Example 10-5c) is, of course, applicable where the roots of Q are all real of multiplicity 1 (as in (7)).

In Example 10-5d since the integrand may be decomposed into the sum of rational functions $\frac{a}{x}$, $\frac{b}{x^2}$, $\frac{cx}{x^2+4}$, $\frac{d}{(\frac{x}{2})^2+1}$ we know that the integral must be of the form stated.

Example TC10-5a. Consider $I = \int \frac{dx}{x^3(x-1)}$. Note that

$$\frac{1}{x^3(x-1)} = \frac{(1-x)(1+x+x^2)+x^3}{x^3(x-1)} = \frac{1}{x-1} - \frac{1}{x^3} - \frac{1}{x^2} - \frac{1}{x} .$$

Thus we composed the integrand into the sum of simpler rational functions which may be integrated at sight:

$$I = \log|x-1| - \log|x| + \frac{1}{x} + \frac{1}{2x^2} + C .$$

Example TC10-5b. The integral $I = \int \frac{2x}{x^2+x+1}\,dx$ is computed by decomposing the integrand into a sum of derivatives of two known functions. We have

$$\frac{2x}{x^2+x+1} = \frac{2x+1}{x^2+x+1} - \frac{1}{x^2+x+1}$$

$$= D\log(x^2+x+1) - D\left(\frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}\right) .$$

Thus $I = \log(x^2+x+1) - \frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + C$.

Solutions Exercises 10-5

1. Integrate the following.

Parts (a) to (k) are simple and an ad hoc approach is probably simpler and quicker than a straight forward application of theory.

(a) $\dfrac{x + 2}{x^2 + 3x + 1}$.

Set $x^2 + 3x + 1 = (x - a)(x - b)$ where

$$a = \frac{-3 + \sqrt{5}}{2} \quad \text{and} \quad b = \frac{-3 - \sqrt{5}}{2} .$$

$$\frac{x + 2}{(x - a)(x - b)} = \frac{(x - a) + 2 + a}{(x - a)(x - b)} = \frac{1}{x - b} + \frac{2 + a}{b - a}\left(\frac{1}{x - b} - \frac{1}{x - a}\right)$$

$$= \frac{2 + b}{b - a}\,\frac{1}{x - b} - \frac{2 + a}{b - a}\,\frac{1}{x - a} ,$$

where

$$\frac{2 + b}{b - a} = \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right) \qquad \frac{2 + a}{b - a} = -\frac{1}{2}\left(1 + \frac{1}{\sqrt{5}}\right) .$$

$$I = \frac{1}{2}\left(1 - \frac{\sqrt{5}}{5}\right)\log\left|x + \frac{3 + \sqrt{5}}{2}\right| + \frac{1}{2}\left(1 + \frac{\sqrt{5}}{5}\right)\log\left|x + \frac{3 - \sqrt{5}}{2}\right| + C .$$

(b) $\dfrac{x^3}{(x^2 + 3x - 10)}$.

By long division,

$$\frac{x^3}{x^2 + 3x - 10} = x - 3 + \frac{19x - 30}{x^2 + 3x - 10} .$$

But

$$\frac{19x - 30}{x^2 + 3x - 10} = \frac{19x - 30}{(x + 5)(x - 2)}$$

$$= \frac{8}{7(x - 2)} + \frac{125}{7(x + 5)} .$$

Consequently,

$$I = \frac{x^2}{2} - 3x + \frac{8}{7}\log|x - 2| + \frac{125}{7}\log|x + 5| + C .$$

(c) $\dfrac{x^3}{x^2 + 2ax + b^2}$, $(b > |a|)$.

$$\frac{x^3}{x^2 + 2ax + b^2} = x - 2a + \frac{(2a^2 - \frac{b^2}{2})[2(x + a)] + (3ab^2 - 4a^3)}{x^2 + 2ax + b^2}$$

$$I = \frac{x^2}{2} - 2ax + (2a^2 - \frac{b^2}{2})\log(x^2 + 2ax + b^2) + \frac{3ab^2 - 4a^3}{\sqrt{b^2 - a^2}} \arctan \frac{x + a}{\sqrt{b^2 - a^2}} + C .$$

(d) $\dfrac{x^2 + \alpha x + \beta}{(x - a)(x - b)}$ (consider the cases $a \neq b$ and $a = b$) .

If $a \neq b$,

$$I = x + \frac{a^2 + \alpha a + \beta}{a - b} \log |x - a| + \frac{b^2 + \alpha b + \beta}{b - a} \log |x - b| + C .$$

If $a = b$,

$$I = x + (\alpha + 2a)\log(x - a) - \frac{a^2 + \alpha a + \beta}{x - a} + C .$$

(e) $\dfrac{x^2}{(x - a)(x - b)(x - c)}$, (a , b ; c distinct).

$$I = \frac{a^2}{(a - b)(a - c)} \log |x - a| + \frac{b^2}{(b - a)(b - c)} \log |x - b|$$
$$+ \frac{c^2}{(c - a)(c - b)} \log |x - c| + C .$$

(f) $\dfrac{x^3 + 1}{x^3 - 1} = 1 + \dfrac{2}{(x - 1)(x^2 + x + 1)} = 1 + \dfrac{2}{3(x - 1)} - \dfrac{2x + 4}{3(x^2 + x + 1)}$.

$$I = x + \frac{2}{3}\log|x - 1| - \frac{1}{3}\log(x^2 + x + 1) - \frac{2}{\sqrt{3}} \arctan \frac{2x + 1}{\sqrt{3}} + C .$$

(g) $\dfrac{1}{x^3 + a^3} = \dfrac{1}{(x + a)(x^2 - ax + a^2)} = \dfrac{1}{3a^2(x + a)} - \dfrac{x - 2a}{3a^2(x^2 - ax + a^2)}$.

$$I = \frac{1}{3a^2} \log|x + a| - \frac{1}{6a^2} \log(x^2 - ax + a^2) - \frac{5\sqrt{3}}{9a^2} \arctan \frac{2x - a}{a\sqrt{3}} + C .$$

(h) $\dfrac{(x+2)^2}{x(x-1)^2} = \dfrac{9}{(x-1)^2} - \dfrac{3}{x-1} + \dfrac{4}{x}$.

$$I = -\frac{9}{x-1} - 3\log|x-1| + 4\log|x| + C .$$

(i) $\dfrac{1}{x^4-1} = \dfrac{1}{4(x-1)} - \dfrac{1}{4(x+1)} - \dfrac{1}{2(x^2+1)}$.

$$I = \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\arctan x + C .$$

(j) $\dfrac{x^2}{x^4-1} = \dfrac{x^2-1}{x^4+1} + \dfrac{1}{x^4-1} = \dfrac{1}{4(x-1)} - \dfrac{1}{4(x+1)} + \dfrac{1}{2(x^2+1)}$

(see (i)).

$$I = \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| + \frac{1}{2}\arctan x + C .$$

(k) $\dfrac{1}{x^4+x^6} = \dfrac{1}{x^4(1+x^2)} = \dfrac{1}{x^4} - \dfrac{1}{x^2} + \dfrac{1}{1+x^2}$.

$$I = -\frac{1}{3x^3} + \frac{1}{x} + \arctan x + C .$$

(ℓ) $\dfrac{x^4}{x^4+1} = 1 - \dfrac{1}{x^4+1}$.

Since $x^4 + 1 > 0$ for all x , $x^4 + 1$ cannot have linear factors. Set

$$(x^4 + 1) = (x^2 + ax + b)(x^2 + cx + d) .$$

Equate coefficients of x^3 to obtain $c = -a$; of x , to obtain $b = d$ ($a = 0$ is not possible), of x^0 to obtain $b = \pm 1$, of x^2 , to obtain $a^2 = \pm 2$, hence only $b = 1$ is possible and $a = \sqrt{2}$.

$$(x^4 + 1) = (x^2 + x\sqrt{2} + 1)(x^2 - x\sqrt{2} + 1) .$$

Set

$$\frac{1}{x^4+1} = \frac{Ax+B}{x^2+x\sqrt{2}+1} + \frac{Cx+D}{x^2-x\sqrt{2}+1} .$$

Use the method of undetermined coefficients to obtain $A = -C = \dfrac{1}{2\sqrt{2}}$, $B = D = \dfrac{1}{2}$; whence,

$$\frac{1}{x^4+1} = \frac{1}{4\sqrt{2}}\,\frac{2x+\sqrt{2}}{x^2+x\sqrt{2}+1} - \frac{1}{4\sqrt{2}}\,\frac{2x-\sqrt{2}}{x^2-x\sqrt{2}+1}$$

$$+\frac{1}{4}\,\frac{1}{x^2+x\sqrt{2}+1} - \frac{1}{4}\,\frac{1}{x^2-x\sqrt{2}+1} .$$

Consequently,

$$I = x - \frac{1}{4\sqrt{2}}\log\left|\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}\right|$$

$$-\frac{1}{2\sqrt{2}}\left[\arctan(1+x\sqrt{2}) + \arctan(1-x\sqrt{2})\right] + C .$$

(m) $\frac{1}{x^6-1}$.

$$x^6-1 = (x^3-1)(x^3+1) = (x-1)(x+1)(x^2+x+1)(x^2-x+1) .$$

$$\frac{1}{x^6-1} = \frac{1}{6}\Big(\frac{1}{x-1} - \frac{1}{x+1} - \frac{x+2}{x^2+x+1} + \frac{x-2}{x^2-x+1}\Big] .$$

$$I = \frac{1}{6}\log\left|\frac{x-1}{x+1}\right| + \frac{1}{12}\log\frac{x^2-x+1}{x^2+x+1} - \frac{2}{\sqrt{3}}\arctan\frac{2}{\sqrt{3}}(x+\tfrac{1}{2})$$

$$-\frac{2}{\sqrt{3}}\arctan\frac{2}{\sqrt{3}}(x-\tfrac{1}{2}) + C .$$

As a special challenge you may wish to ask for $I = \int \frac{dx}{1+x^6}$ which Leibniz failed to represent in elementary terms. For this, note that

$$1+x^6 = (1+x^2)(1-x^2+x^4)$$

$$= (1+x^2)(1-x\sqrt{3}+x^2)(1+x\sqrt{3}+x^2) .$$

In this case,

$$I = \frac{\arctan x}{3} + \frac{1}{4\sqrt{3}}\log\frac{x^2+x\sqrt{3}+1}{x^2-x\sqrt{3}+1} + \frac{1}{6}\arctan(2x+\sqrt{3})$$

$$+\frac{1}{6}\arctan(2x-\sqrt{3}) + C .$$

2. Prove from Equation (3) that if $Q(x) = (x - a_1)(x - a_2) \dots (x - a_n)$, where $a_1 < a_2 < \dots < a_n$, then $\frac{1}{Q(x)}$ has a decomposition into partial fractions of the form

$$\frac{1}{Q(x)} = \frac{r_1}{x - a_1} + \frac{r_2}{x - a_2} + \dots + \frac{r_n}{x - a_n} .$$

From Equation (3),

$$\frac{1}{Q(x)} = \frac{1}{a_1 - a_2}\left(\frac{1}{x - a_1} - \frac{1}{x - a_2}\right)\frac{1}{(x - a_3) \dots (x - a_n)} .$$

Thus $\frac{1}{Q(x)} = \left(\frac{1}{P_1(x)} - \frac{1}{P_2(x)}\right)\frac{1}{a_1 - a_2}$ where P_1 and P_2 are each products of $n - 1$ linear factors. Repeat the process, reducing the numbers of factors in each denominator by 1 at each step. After $n - 1$ steps, collect terms.

3. Prove if

$$a_nx^n + a_{n-1}x^{n-1} + \dots + a_0 = b_nx^n + b_{n-1}x^{n-1} + \dots + b_0$$

for all but finitely many numbers x, that the coefficients of like powers on the right and left are equal; i.e., $a_k = b_k$ for $k = 0, 1, \dots, n$.

Consider $P(x) = \sum_{k=0}^{n} (a_k - b_k)x^k$. Since $P(x) = 0$ for more than n numbers x it follows from Corollary 2 to Lemma 5-3 that all the coefficients $a_k - b_k$ of P must vanish; otherwise a polynomial of degree less than or equal to n would have more than n roots.

4. Verify that $\int \frac{px + q}{[(x - a)^2 + b^2]}\,dx$, $b > 0$, can be expressed as the sum of terms of the forms (11a, b, c).

As in the text, substitute $x = a + b \tan \theta$.

$$I = \frac{pa + q}{b} \arctan \frac{x - a}{b} + \frac{p}{2} \log [(x - a)^2 + b^2] + C .$$

TC10-6. Definite Integrals.

Unless you call it to his attention, the student is likely to miss one of the important applications of the validity of the Substitution Rule (Property d). Namely, if an improper integral I can be transformed by an invertible substitution into a form which is known to converge, then I itself converges. For example, by the substitution $x = \frac{1}{u}$ we transform the improper integral

$$I = \int_{1}^{\infty} \frac{1}{x^2}\, dx \ ;$$

into the regular form

$$\int_{0}^{1} du \ ,$$

ipso facto convergent. Hence by the inverse substitution, $x = \frac{1}{u}$, I itself is convergent.

The theorems of integral calculus hold for convergent improper integrals only. It is senseless to speak of "properties" of divergent improper integrals; divergent integrals are not numbers. Yet, we have to write the symbolic integral first in order then to decide whether it is convergent and meaningful or divergent and meaningless. Only after we have decided that it is a convergent integral can we justify the manipulations by which we compute it. When a given integral cannot be evaluated by computing antiderivatives, we need rules or criteria to ascertain whether a given integral converges or diverges. Sometimes it is more convenient to manipulate the integral first and check the reliability of our results afterward (as in Example 10-6g). We note that the definition of "imporper integral" does not imply the integral exists. It calls attention to possible nonexistence.

In the discussion of Example 10-6g we allude to the possibility of extending the definition of the factorial function from the domain of nonnegative integers to a function on the domain of all nonnegative real numbers. Surprisingly, it was only as recently as 1922 that a proof of uniqueness of this extension under simple general conditions was first presented by H. Bohr and J. Mollerup. Euler's gamma function, given by

$$\Gamma(x) = \int_{0}^{\infty} t^{x-1}\, e^{-t}\, dt \ ,$$

is the only function defined and positive for $x > 0$, which satisfies

(a) $\Gamma(1) = 1$,

(b) $\Gamma(x + 1) = x\,\Gamma(x)$,

(c) $y = \log \Gamma(x)$ is flexed upward.

Conditions (a) and (b) yield the relation for integer points,

$$\Gamma(n) = (n - 1)! \ .$$

The proof of the Bohr-Mollerup Theorem given in E. Artin, The Gamma Function (Hold, Reinhart and Winston, 1964), is within the grasp of the student who completes Chapter 14.

Note that Definition 10-6 extends the definition of partition to unbounded intervals.

The integral

$$\int_0^1 \frac{dx}{x}$$

is divergent. If the fact were not noted and we treated this symbol as a real number I , then we would have

$$\begin{aligned} I = \int_0^1 \frac{dx}{x} &= \int_0^a \frac{dx}{x} + \int_a^1 \frac{dx}{x} \\ &= \int_0^a \frac{dx}{x} - \log a \ . \end{aligned}$$

Then, employing the substitution $x = at$, $dx = a\,dt$, we would obtain

$$\int_0^a \frac{dx}{x} = \int_0^1 \frac{dt}{t} = I \ .$$

Thus we would conclude that $I = I - \log a$ and obtain the paradoxical result $\log a = 0$ for all positive a .

Example T.C. 10-6. The indefinite integral of $\log \sin x$ is not an elementary function, but the integral over $[0, \frac{\pi}{2}]$ can be evaluated by a special device. Set

$$I = \int_0^{\pi/2} \log \sin x \, dx$$

and use the substitution $x = \frac{\pi}{2} - t$, $dx = -dt$ to obtain

$$I = \int_0^{\pi/2} \log \cos x \, dx \ .$$

Add the two represntations to obtain

(a)
$$2I = \int_0^{\pi/2} \log(\sin x \cos x)dx$$
$$= \int_0^{\pi/2} [\log \sin 2x - \log 2]dx \ .$$

Now, set $x = \frac{u}{2}$, $dx = \frac{du}{2}$ to obtain

(b)
$$\int_0^{\pi/2} \log \sin 2x \ dx = \frac{1}{2}\int_0^{\pi} \log \sin x \ dx \ .$$

However, from the substitution $x = \pi - v$,

(c)
$$\int_{\pi/2}^{\pi} \log \sin x \ dx = \int_0^{\pi/2} \log \sin v \ dv = I \ .$$

Enter the results of (b) and (c) in (a) to obtain

$$2I = I - \int_0^{\pi/2} \log 2 \ dx \ ,$$

whence,

$$I = -\frac{\pi}{2} \log 2 \ .$$

In order to appreciate some of the theoretical issues involved, even those students who are not expected to attempt the Exercises 10-6b, Numbers 1, 5, 16, 17, 18, should read them and feel free to apply them as needed.

Exercises 10-6b, Number 7, illustrates an interesting (and remarkable) geometric fact. For the function $f : x \longrightarrow x^r$, if $-1 < r < -\frac{1}{2}$, the area of the standard region of f over $[0,1]$ is finite but the volume of the solid obtained by rotating this standard region about the x-axis is infinite. On the other hand, the area of the standard region of $f : x \longrightarrow \frac{1}{x}$ over $[1,\infty)$ is infinite, whereas the volume of the solid obtained by revolving the standard region about the x-axis is finite.

Solutions Exercises 10-6a

1. $$\int_{-99}^{99} \frac{\sin^{99} \frac{x}{99}}{x^2 + 99^2} dx \ .$$

$I = 0$; the integrand is odd.

2. $\int_0^1 x^3\, e^{-3x^2}\, dx$.

Substitute $u = 3x^2$.

$$I = \frac{1}{18}\int_0^3 ue^{-u}\, du = -\frac{1}{18}(u + 1)e^{-u}\Big|_0^3 = \frac{1}{18}\left(1 - \frac{4}{e^3}\right) .$$

3. $\int_1^e \log^3 x\, dx$.

Integrate by parts with $dv = dx$, $u = \log^k x$, repeatedly, to obtain

$$I = 2(3 - e) .$$

4. $\int_0^{\pi/2} \sin^m x\, dx$.

From Exercises 10-4, Number 4(a),

$$I_m = \frac{m - 1}{m} I_{m-2} .$$

Since $I_0 = \frac{\pi}{2}$ and $I_1 = 1$, if m is even, $I = \frac{\pi}{2}\,\frac{1\cdot 3\cdot 5\cdots m - 1}{2\cdot 4\cdot 6\cdots m}$; if m is odd, $I = \frac{2\cdot 4\cdot 6\cdots m - 1}{3\cdot 5\cdot 7\cdots m}$.

5. $\int_0^{\pi/2} \sin^m x \cos^m x\, dx$, (m , a positive integer).

$$I = \int_0^{\pi/2} \frac{\sin^m 2x}{2^m}\, dx$$

$$= \frac{1}{2^{m+1}}\int_0^{\pi} \sin^m \theta\, d\theta$$

where $\theta = 2x$. As in Number 4, $I_m = \frac{m - 1}{m} I_{m-2}$; but here $I_0 = \pi$ and $I_1 = 2$.

6. $\int_0^{\pi/2} \frac{dx}{a + b \cos x}$, $a > b \geq 0$.

Set $x = 2 \arctan t$, $\cos x = \frac{1 - t^2}{1 + t^2}$, $dx = \frac{2\,dt}{1 + t^2}$.

$$I = \int_0^1 \frac{2}{(a + b) + (a - b)t^2}\,dt = \frac{2}{\sqrt{a^2 - b^2}} \arctan \sqrt{\frac{a - b}{a + b}} .$$

7. $\int_0^{\pi/2} \sin^7 x \cos^3 x\,dx$.

$$I = \int_0^{\pi/2} \sin^7 x\,(1 - \sin^2 x)\cos x\,dx = \frac{1}{8} - \frac{1}{10} = \frac{1}{40} .$$

8. $\int_1^2 \frac{dx}{x + x^5}$.

$$I = \int_1^2 \frac{x^{-5}}{1 + x^{-4}}\,dx ,$$ (See Exercises 10-2, No. 1(f)).

Set $t = x^{-4}$. Then

$$I = \frac{1}{5} \int_{1/16}^1 \frac{dt}{1 + t} = \frac{1}{5} \log \frac{32}{17} .$$

9. $\int_0^b \sqrt{b^2 - x^2}\,dx = b^2 \int_0^{\pi/2} \cos^2 t\,dt = \frac{\pi}{4} b^2$ where $t = b \sin x$.

10. $\int_{-\pi/4}^{\pi/4} \frac{\sin^5 \theta + 1}{a^2 \sin^2 \theta + b^2 \cos^2 \theta}\,d\theta$, $a > 0$, $b > 0$.

$$I = \int_{-\pi/4}^{\pi/4} \frac{1}{a^2 \sin^2 \theta + b^2 \cos^2 \theta}\,d\theta$$

since $\frac{\sin^5 \theta}{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$ is odd. Set $t = \tan \theta$ (compare Exercises 10-2, No. 2(i)), to obtain

$$I = \int_{-1}^1 \frac{dt}{b^2 + a^2 t^2} = \frac{2}{ab} \arctan \frac{a}{b} .$$

11. Compare $\int_0^a f(x)dx$ with $\int_{-a}^0 f(x)dx$ when f is even or odd to derive the results (1) and (2) of the text by a method other than the one you employed for Exercises 6-4, Number 4.

Substitute $x = -t$ in $\int_0^a f(x)dx$ to obtain

$$\int_0^a f(x)dx = \int_{-a}^0 f(-x)dx .$$

Hence,

$$\int_{-a}^a f(x)dx = \int_{-a}^0 f(x)dx + \int_0^a f(x)dx$$

$$= \begin{cases} 0 \text{ , if } f \text{ is odd,} \\ 2\int_0^a f(x)dx \text{ , if } f \text{ is even.} \end{cases}$$

12. Prove if f is integrable and periodic of period p, then for all a and b

$$\int_a^{a+p} f(x)dx = \int_b^{b+p} f(x)dx .$$

Follow the geometrical approach of the text. Set $k = 1 + \left[\!\left[\frac{b-a}{p}\right]\!\right]$. Then $b < a + kp \leq b + p$. Consequently,

$$I = \int_b^{b+p} f(x)dx = \int_b^{a+kp} f(x)dx + \int_{a+kp}^{b+p} f(x)dx .$$

Now in the integral from b to $a + kp$ make the substitution $u = x - (k - 1)p$; in the integral from $a + kp$ to $b + p$, $u = x - kp$. Then

$$I = \int_{b-(k-1)p}^{a+p} f(u + kp)du + \int_a^{b-(k-1)p} f(u + (k - 1)p)du$$

$$= \int_{b-(k-1)p}^{a+p} f(u)du + \int_a^{b-(k-1)p} f(u)du$$

$$= \int_a^{a+p} f(u)du .$$

by Theorem 6-4b.

13. Prove that if $n \geq 2$ then

$$.500 < \int_0^{1/2} \frac{dt}{\sqrt{1 - t^n}} < .524 .$$

If $0 \leq t \leq 1$, then $0 \leq t^n \leq t^2 \leq 1$ and $0 \leq 1 - t^2 \leq 1 - t^n \leq 1$. Thus

$$.500 = \int_0^{1/2} dt < \int_0^{1/2} \frac{dt}{\sqrt{1 - t^n}} \leq \int_0^{1/2} \frac{dt}{\sqrt{1 - t^2}}$$

$$\leq \arcsin \frac{1}{2} \leq \frac{\pi}{6} < .524 .$$

14. Prove that $\int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x} dx = \pi^2$.

Since $\frac{x}{1 + \cos^2 x}$ is odd; $\frac{x \sin x}{1 + \cos^2 x}$, even,

$$I = 4 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = 4 \int_0^{\pi} \frac{x \sin x}{2 - \sin^2 x} dx$$

$$= 2\pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx ,$$

by Exercises 7-3, Number 12. Hence,

$$I = -2\pi \arctan \cos x \Big|_0^{\pi} = -2\pi(- \frac{\pi}{4} - \frac{\pi}{4}) = \pi^2 .$$

15. Show $\frac{2^2}{1 \cdot 3} \cdot \frac{4^2}{3 \cdot 5} \cdot \frac{6^2}{5 \cdot 7} \cdots \frac{(2n)^2}{(2n - 1)(2n + 1)} = \frac{1}{2n + 1} \left[\frac{2^{2n}(n!)^2}{(2n)!}\right]^2$.

First, observe that $2 \cdot 4 \cdot 6 \cdots (2n) = 2^n(n!)$; then that

$$1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n - 1) = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2n)}{2 \cdot 4 \cdot 6 \cdots (2n)} = \frac{2n!}{2^n(n!)} .$$

16. Determine the value exact to three deciaml places of

$$\int_1^{e^{36.1}} \frac{\sin(\pi \log x)}{x}\, dx\ .$$

Set $\theta = \pi \log x$.

$$I = \frac{1}{\pi}\int_0^{36.1\pi} \sin\theta\, d\theta = -\frac{1}{\pi}\cos\theta\ \Big|_0^{\pi/10}$$

where $\int_0^{2\pi} \sin\theta\, d\theta = 0$ is used. From Example 7-5b

$$\cos\frac{\pi}{10} = 1 - \frac{\pi^2}{200} + \epsilon$$

where $0 < \epsilon \leq \frac{1}{24}\left(\frac{\pi}{10}\right)^4$. Since $\pi^2 < 10$, the error term may be neglected to the desired accuracy. Hence, to the nearest thousandth,

$$I = \frac{\pi}{200} = .016\ .$$

17. Evaluate $\int_{-\pi/4}^{\pi/4} \frac{t + \frac{\pi}{4}}{2 - \cos 2t}\, dt$.

Observe that $\frac{t}{2 - \cos 2t}$ is odd; hence

$$I = \frac{\pi}{4}\int_{-\pi/4}^{\pi/4} \frac{dt}{2 - \cos 2t}\ .$$

Set $u = \tan t$, $\cos 2t = \frac{1 - \tan^2 t}{1 + \tan^2 t} = \frac{1 - u^2}{1 + v^2}$, $dt = \frac{du}{1 + u^2}$. Then

$$I = \frac{\pi}{2}\int_0^1 \frac{du}{1 + 3u^2} = \frac{\pi}{2\sqrt{3}}\arctan u\sqrt{3}\ \Big|_0^1 = \frac{\pi^2}{6\sqrt{3}}\ .$$

Solutions Exercises 10-6b

1. (a) Let f be Riemann integrable over $[a,b]$. Show that the integral of f in the sense of (1) exists and is equal to the Riemann integral.

Let M be an upper bound for $f(x)$ on $[a,b]$ and set $I = \int_a^b f(t)dt$. Then for $a < \alpha \leq \beta < b$,

$$|I - \int_\alpha^\beta f(t)dt = \left|\int_a^\alpha f(t)dt + \int_b^\beta f(t)dt\right| \leq M[(\alpha - a) + (b - \beta)]$$

from which the result follows straightforwardly.

(b) Show that if f is continuous on (a,b) and the limits $\lim_{x\sim a^+} f(x)$ and $\lim_{x\sim b^-} f(x)$, exist, then $\int_a^b f(x)dx$ exists in the sense of (1).

The integral in the sense of (1) does not depend on the values of f at the endpoints. The function f^* which has the same values as f on (a,b) and is defined at the endpoints by $f^*(a) = \lim_{x\sim a^+} f(x)$ and $f^*(b) = \lim_{x\sim b^-} f(x)$ is continuous hence integrable over $[a,b]$. Apply Part (a) to obtain the result.

Note that ordinary integrability does not depend on how f^* is defined at a and b, just that it is defined.

2. Show that the basic theorems for Riemann integrals hold also for integrals in the extended sense:

(a) If f and g are integrable over $[a,b]$ in the extended sense, and $f(x) \leq g(x)$ whenever both are defined, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

(Theorem 6-4a).

Let $\{x_0,x_1,\ldots,x_n\}$ be a partition of $[a,b]$ which includes the discontinuities of both f and g. Let ξ_i be any point of (x_{i-1},x_i). For $x_{i-1} < \alpha_i < \beta_i < x_i$, by Theorem 6-4a,

$$\int_{\alpha_i}^{\xi_i} f(t)dt \leq \int_{\alpha_i}^{\xi_i} g(t)dt$$

and

$$\int_{\xi_i}^{\beta_i} f(t)dt \leq \int_{\xi_i}^{\beta_i} g(t)dt .$$

From Theorem 3-4 it follows as α_i approaches x_{i-1}^{+} and β_i approaches x_i^{-} that the inequalities hold in the limit. Add the two limits to obtain by (1)

$$\int_{x_{i-1}}^{x_i} f(t)dt \leq \int_{x_{i-1}}^{x_i} g(t)dt$$

and sum from $i = 1$ to n to obtain the desired result.

(b) If a , b and c are points of an interval over which f is integrable in the extended sense, then

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

(Corollary to Theorem 6-4b).

By Number 17, if a , b , and c are not partition points we may add them to the partition of Definition 10-6. Now, if $a < b < c$, say $a = x_m$, $b = x_{m+p}$, $c = {}_{m+p+q}$, then by Definition 10-6,

$$\int_a^b f(t)dt + \int_b^c f(t)dt = \sum_{i=m+1}^{m+p} \int_{x_{i-1}}^{x_i} f(t)dt + \sum_{i=m+p+1}^{m+p+q} \int_{x_{i-1}}^{x_i} f(t)dt$$

$$= \sum_{i=m+1}^{m+p+q} \int_{x_{i-1}}^{x_i} f(t)dt$$

$$= \int_a^b f(t)dt \ .$$

The proof for any other ordering of a , b , and c follows in the same way as the Corollary to Theorem 6-4b (Exercises 6-4, No. 1).

(c) If f and g are integrable over $[a,b]$ in the extended sense, then

$$\int_a^b [\alpha f(x) + \beta g(x)] = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$$

(Theorem 6-4c).

Let $\{x_0,x_1,\ldots,x_n\}$ be a partition of $[a,b]$. For $x_{i-1} < \xi < x_i$,

$$\alpha \int_{x_{i-1}}^{x_i} f(t)dt + \beta \int_{x_{i-1}}^{x_i} g(x)dx$$

$$= \alpha \lim_{u \to x_{i-1}^+} \int_u^\xi f(t)dt + \beta \lim_{u \to x_{i-1}^+} \int_u^\xi g(t)dt$$

$$+ \alpha \lim_{v \to x_i^-} \int_\xi^v f(t)dt + \beta \lim_{v \to x_i^-} \int_\xi^v g(t)dt$$

(From (1))

$$= \lim_{u \to x_{i-1}^+} \int_u^\xi [\alpha f(t) + \beta g(t)]dt + \lim_{v \to x_i^-} \int_\xi^v [\alpha f(t) + \beta g(t)]dt$$

(Corollary to Theorem 3-4c, Theorem 6-4c)

$$= \int_{x_{i-1}}^{x_i} [\alpha f(t) + \beta g(t)]dt \qquad \text{(From (1)).}$$

Sum from $i = 1$ to $i = n$ to obtain the desired result.

(d) Let f be continuous on (a,b) and integrable over $[a,b]$. On the domain consisting of an open interval I with endpoints α and β (we permit $\alpha > \beta$), let g be continuous1y differentiable. If the range of g is in (a,b), and $\lim_{t \to \alpha} g(t) = a$ and $\lim_{t \to \beta} g(t) = b$, then

$$\int_\alpha^\beta fg(t)g'(t)dt = \int_a^b f(x)dx .$$

(Substitution Rule, Theorem 10-2.) The proof requires the demonstration of existence for the integral on the left.

For this purpose it is convenient to introduce the concepts "neighborhood of infinity" and "neighborhood of minus infinity." A *neighborhood of ∞* is an open ray of the form $\{x : x > a\}$. Similarly, a *neighborhood of $-\infty$* is a ray of the form $\{x : x < a\}$. For a neighborhood of ∞ or $-\infty$, the neighborhood and the deleted neighborhood are the same. Furthermore a neighborhood of ∞ is a left-sided neighborhood; a neighborhood of $-\infty$ is a right-sided neighborhood. We now extend the meaning of

$$\lim_{x \to a} f(x) = b ,$$

so that a and b may be not real numbers but ∞, or $-\infty$; that is, every deleted neighborhood of a contains points of the domain of f and for each neighborhood J of b there exists a deleted neighborhood I of a wherein f maps the points of its domain into J.

For $a < z < b$ we have

$$\text{(i)} \quad \int_a^b f(x)dx = \lim_{r\sim a^+}\int_r^z f(x)dx + \lim_{s\sim b^-}\int_z^s f(x)dx \qquad \text{(From (1))}.$$

$$\text{(ii)} \quad = \lim_{u\sim\alpha}\int_u^\zeta fg(t)g'(t)dt + \lim_{v\sim\beta}\int_\zeta^v fg(t)g'(t)dt$$

(See below)

$$= \int_\alpha^\beta fg(t)g'(t)dt \qquad \text{(From (1))} .$$

Here $g(\zeta) = z$; such a point ζ exists by the Intermediate Value Theorem.

For the proof it is necessary to show that the limits in line (i) exist and are equal to the corresponding limits in (ii). Set $x = g(t)$. For v in (α,β) , $s = g(v)$ we have by the Substitution Rule

$$\begin{aligned}\phi(s) &= \int_z^s f(x)dx \\ &= \int_\zeta^v fg(t)g'(t)dt \\ &= \phi g(v) .\end{aligned}$$

Now, by hypothesis, $B = \lim_{s\sim b^-}\phi(s)$ exists (B is real since the integral is convergent). Consequently given a neighborhood K of B , ϕ maps some left-sided neighborhood J of b into K . Furthermore, since $\lim_{v\sim\beta} g(v) = b$ (in the extended sense of limit) and the range of g lies on the left side of b , it follows that there is a neighborhood I of β wherein g maps the point of its domain into J . Thus ϕg , which has the same domain as g , maps those points of its domain within I into K . In other words, ϕg has a limit at β and

$$\lim_{v\sim\beta}\phi(r) = \lim_{s\sim b^-}\phi(s) .$$

Similarly,

$$\lim_{u\sim\alpha}\phi g(v) = \lim_{r\sim a^+}\phi(r) .$$

This completes the proof of line (b).

3. Prove the corollary to Theorem 10-6a.

The corollary asserts, if h is Riemann integrable on every closed subinterval on (a,b) and $\int_a^b h(x)$ diverges, then if $f(x) \geq h(x) \geq 0$, $\int_a^b f(x)dx$ diverges. If $\int_a^b f(x)dx$ were to converge, then by Theorem 10-6a so would $\int_a^b h(x)dx$ in contradiction to the hypothesis.

4. Prove Theorem 10-6b when $a > b$.

We have $\int_\alpha^\beta \frac{dx}{(x - \alpha)^p}$, $(\alpha < \beta)$, converges if $p < 1$ and diverges if $p \geq 1$. Now

$$\int_a^b \frac{dx}{|x - a|^p} = - \int_{-a}^{-b} \frac{du}{|u + a|^p}$$

where $u = -x$. Set $-a = \alpha$, $-b = \beta$ to obtain

$$- \int_\alpha^\beta \frac{du}{(u - \alpha)^p}$$

where $\beta > \alpha$. If this last integral converges $(p < 1)$, then apply the inverse substitution and the result of Number 2(d) to prove the convergence of the original integral; if the last integral diverges $(p \geq 1)$, then the original integral cannot converge for the substitution of u for x would otherwise have to yield a convergent integral.

5. Prove that $\int_0^a \frac{A}{|x - a|^p} dx$ converges if $p < 1$ and diverges if $p \geq 1$.

From Theorem 10-6b, for $p < 1$ $\int_0^a \frac{1}{|x - a|^p} dx$ converges. It follows from the result of Number 2(c) that

$$\int_0^a \frac{A}{|x - a|^p} dx = A \int_0^a \frac{dx}{|x - a|^p}$$

converges. Conversely, if $A \neq 0$ and $p \geq 1$ then $\int_0^a \frac{A}{|x - a|^p}\,dx$ diverges, for if the integral converged, then so would

$$\frac{1}{A}\int_0^a \frac{A}{|x - a|^p}\,dx = \int_0^a \frac{1}{|x - a|^p}\,dx \ .$$

6. Prove Theorem 10-6c.

$$\int_a^{\xi} \frac{dx}{x^p} = \begin{cases} \dfrac{\xi^{1-p} - a^{1-p}}{1 - p}\ , & p \neq 1 \\ \log \xi - \log a\ , & p = 1\ ,\ (a > 0)\ . \end{cases}$$

Take the limit as ξ approaches ∞ .

7. Test for convergence and divergence,

$$I = \int_0^{\infty} x^r\,dx\ .$$

I always diverges. If $r \geq -1$ then I diverges at ∞ by Theorem 10-6c, and if $r \leq -1$, then I diverges at 0 by Theorem 10-6b.

8. Evaluate those of the following improper integrals which converge.

Where used below, I denotes the integral of the stated problem.

(a) $\int_{-2}^{4} \frac{dx}{\sqrt[5]{(4 - x)^3}}$. $\frac{5 \cdot 6^{2/5}}{2}$

(b) $\int_0^{a^{1/m}} \frac{x^{m-1}\,dx}{\sqrt{a^2 - x^{2m}}}$, $m > 0$.   Substitute $u = \frac{x^m}{a}$. $\frac{\pi}{2m}$.

(c) $\int_0^1 \log^2 x\,dx$.   Compare Exercises 10-4, No. 1e . 2 .

(d) $\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \cos\theta}}$.   Diverges. Use $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$ to obtain $\sqrt{1 - \cos\theta} = \sqrt{2}\sin\frac{\theta}{2}$ and note as in Example 10-6j that $\sin\frac{\theta}{2} \geq \frac{\theta}{\pi}$.

(e) $\int_0^{\infty} (ax + b)^2 e^{-x}\,dx$.   Use Example 10-6g. $2a^2 + 2ab + b^2$.

(f) $\int_1^\infty x^3 e^{-2x^2}\,dx$.   Substitute $x^2 = t$. $\frac{3}{8e^2}$.

(g) $\int_0^\infty x^{n-1} e^{-ax}\,dx$; $a > 0$; n , natural number.

Substitute $t = ax$. $\frac{(n-1)!}{a^n}$.

(h) $\int_0^\infty \frac{dx}{x\sqrt{1+x^3}}$

Set $x^3 = t^3 - 1$, $x^2dx = t^2dt$.

$$I = \int_{\sqrt[3]{2}}^\infty \frac{t}{t^3-1}\,dt = \frac{1}{3}\log(t-1) - \frac{1}{6}\log(t^2+t+1) + \frac{1}{\sqrt{3}}\arctan\frac{2t+1}{\sqrt{3}}\Bigg|_{\sqrt[3]{2}}^\infty$$

$$= \frac{1}{6}\log\frac{(t-1)^3}{t^3-1} + \frac{1}{\sqrt{3}}\arctan\frac{2t+1}{\sqrt{3}}\ \Bigg|_{\sqrt[3]{2}}^\infty$$

$$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} - \arctan\frac{2^{4/3}+1}{\sqrt{3}}\right) - \frac{1}{2}\log(2^{1/3}-1) .$$

(i) $\int_1^\infty \sqrt[3]{\frac{x-1}{x+1}}\,\frac{dx}{x}$.

Divergent. For $x > \frac{9}{7}$, $\frac{x-1}{x+1} > \frac{1}{8}$ and $\sqrt[3]{\frac{x-1}{x+1}} > \frac{1}{2}$; th[illegible]
$\frac{1}{x}\sqrt[3]{\frac{x-1}{x+1}} > \frac{2}{x}$. Apply Theorem 10-6c.

(j) $\int_a^b \frac{dx}{\sqrt{(b-x)(x-a)}}$, $a < b$.

$(b-x)(a-x) = \left(\frac{b-a}{2}\right)^2 - \left(x - \frac{a+b}{2}\right)^2$. Set
$x = \frac{a+b}{2} + \frac{b-a}{2}\sin t$. Then $I = \int_{-\pi/2}^{\pi/2} dt = \pi$.

9. **Determine whether or not the following integrals are convergent and evaluate when practical.**

Let $f(x)$ denote the integrand and I , the integral.

(a) $\displaystyle\int_1^{\infty} \frac{x^3 - 1}{x^4 + 1}\, dx$.

Divergent. Integrand behaves like $\frac{A}{x}$ for large x :

$$f(x) = \frac{1}{x}\left(\frac{1 - \frac{1}{x^3}}{1 + \frac{1}{x^4}}\right) = \frac{h(x)}{x}$$

where $\lim\limits_{x \to \infty} h(x) = 1$. Thus, for sufficiently large x , $h(x)$ is bounded away from 0 . For $x > \sqrt[3]{2}$, $h(x) > \frac{1}{2}(1 - \frac{1}{x^3}) > \frac{1}{4}$.

(b) $\displaystyle\int_1^{\infty} \frac{\log(x^2 + 1)}{x^{3/2}}\, dx$.

Convergent. From Lemma 8-3, for any positive α , $\lim\limits_{u \to \infty} \frac{\log u}{u^{\alpha}} = 0$.

Hence, for sufficiently large x ,

$$\log(x^2 + 1) < \log x^3 < 3 \log x < 3x^{1/4}$$

and

$$f(x) < \frac{3x^{1/4}}{x^{3/2}} < \frac{3}{x^{5/4}}$$

so that I converges by Theorem 8-6c.

If specific estimates of "sufficiently large x" are desired use the inequality $\log u < u$, as follows. For $x > 2$

$$\log(x^2 + 1) < \log 2x^2 < \log x^3 \leq 12 \log x^{1/4} < 12x^{1/4} .$$

Thus $f(x) < \frac{12}{x^{5/4}}$.

(c) $\displaystyle\int_1^e \frac{dt}{t\sqrt{\log t}}$. Set $\log t = x$. $I = 2\sqrt{2}$.

(d) $\displaystyle\int_0^2 \frac{dt}{e^{\sqrt{t}} - 1}$.

Converges. From Section 8-6, Formula (3),

$$e^{\sqrt{t}} - 1 \quad (1 + \sqrt{t}) - 1 \quad \sqrt{t} \ ;$$

thus $\displaystyle\frac{1}{e^{\sqrt{t}} -1} < \frac{1}{\sqrt{t}}$

(e) $\displaystyle\int_0^1 \log^n x \, dx$. Converges. Set $x = e^{-u}$, then

$$I = (-1)^n \int_0^\infty u^n e^{-u} \, du = n!(-1)^n \ .$$

(f) $\displaystyle\int_0^\infty e^{-u} \log^n u \, du$.

Converges. For $u \in (0,1)$,

$$|e^{-u} \log^n u| < |\log^n u|$$

which yields convergence at 0 by comparison with Part (e). For $u \in (1,\infty)$, $\log u < u$ and

$$e^{-u} \log^n u < u^n e^{-u}$$

which is convergent by comparison with Example 10-6g.

(g) $\displaystyle\int_0^\infty \frac{x^2 - 1}{(x^2 + 1)\sqrt{1 + x^4}} \, dx$. Convergent.

$$\frac{x^2 - 1}{(x^2 + 1)\sqrt{1 + x^4}} < \frac{1}{\sqrt{1 + x^4}} < \frac{1}{x^2} \ .$$

(h) $\int_{-1}^{\infty} \frac{x^2 dx}{\sqrt{|x^7 + x^3 + 1|}}$.

Convergent. Set $P(x) = x^7 + x^3 + 1$. Since $P'(x) = 7x^6 + 3x^2 \geq 0$, P is increasing. Since $P(-1) = -1$, $P(0) = 1$, it follows that P has its unique root α in $(0,1)$. The root α is simple, for if $P(x) = (x - \alpha)^2 R(x)$ then $P'(\alpha) = (x - \alpha)[R'(\alpha) + 2(x - \alpha)R(\alpha)] = 0$, but P' has its only root at 0 . Thus $P(x) = (x - \alpha)Q(x)$ where $Q(\alpha) \neq 0$. It follows that $Q(x) > 0$ on $(-1,\infty)$. From the continuity of Q , there is a δ-neighborhood of α within $(-1,0)$ where $Q(x) > \frac{1}{2} Q(\alpha)$, (Corollary 1 to Lemma 3-4). Thus on $(\alpha - \delta , \alpha + \delta)$

$$\frac{x^2}{\sqrt{|P(x)|}} = \frac{x^2}{\sqrt{|x - \alpha|Q(x)}} \leq \sqrt{\frac{2}{Q(\alpha)}} \frac{1}{\sqrt{|x - \alpha|}}$$

and convergence follows from Theorem 8-6b.

(i) $\int_{0}^{\infty} \frac{dx}{\sqrt{2x^4 - 2x^3 - x^2 + 1}}$.

Diverges. $2x^4 - 2x^3 - x^2 + 1 = (x - 1)^2(2x^2 + 2x + 1)$. But $2x^2 + 2x + 1$ is increasing on $(0,\infty)$; hence on $(0,1)$, $2x^2 + 2x + 1 < 5$. It follows that on $(0,1)$

$$f(x) \geq \frac{1}{(1 - x)\sqrt{5}}$$

and hence by Theorem 8-6b, the integral diverges.

(j) $\int_{0}^{1} \frac{dx}{\sqrt{1 - x^4}}$.

Converges. On $(0,1)$,

$$(1 - x^4) = (1 - x)(1 + x + x^2 + x^3) \geq 1 - x ,$$

hence,

$$f(x) \leq \frac{1}{\sqrt{1 - x}} .$$

(k) $\displaystyle\int_0^{\pi} \frac{\sin x}{\sqrt{1 - \cos x}}\, dx$.

Converges. From $\cos x = 1 - 2 \sin^2 \frac{x}{2}$,

$$f(x) = \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{\sqrt{2} \sin \frac{x}{2}} = \sqrt{2} \cos \frac{x}{2} ,$$

whence $I = 2\sqrt{2}$.

(ℓ) $\displaystyle\int_0^{\pi} \frac{\cos x}{\sqrt{1 - \cos x}}\, dx$.

Diverges. From $\cos x = 1 - 2 \sin^2 \frac{x}{2}$, $f(x) = \dfrac{\cos x}{\sqrt{2} \sin \frac{x}{2}}$.

However, for all $u > 0$ $\sin u < u$. Consequently, for $x \in (0, \frac{\pi}{4})$ $f(x) > \dfrac{\sqrt{2}/2}{\sqrt{2}\, x/2} \geq \dfrac{1}{x}$.

10. Consider

$$I = \int_{-\infty}^{\infty} \sqrt{|R(x)|}\, dx$$

where $R(x)$ is a rational function, $R(x) = \dfrac{P(x)}{Q(x)}$, where the polynomials $P(x)$ and $Q(x)$ have no common factors and $Q(x)$ has only simple roots. Determine the conditions under which I converges or diverges.

Let α be any real root of $Q(x)$. Under the conditions of the hypothesis,

$$|R(x)| = \frac{1}{|x - \alpha| S(x)}$$

where $S(\alpha) \neq 0$. In a neighborhood of $S(\alpha)$ then,

$$|R(x)| \leq \frac{2/S(\alpha)}{|x - \alpha|} .$$

Thus the integral converges at the zeros of $Q(x)$. Difficulties can only arise at the ends of integration. Suppose

$$R(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_0}$$

where a_n and b_m are not zero. Observe for $x > 0$ that

$$1 - \left|\sum_{\nu=0}^{n-1} \frac{a_\nu x^\nu}{a_n x^n}\right| \leq 1 - \sum_{\nu=0}^{n-1} \left|\frac{a_\nu}{a_n x^{n-\nu}}\right| \leq \frac{|P(x)|}{|a_n x^n|}$$

$$\leq 1 + \sum_{\nu=0}^{n-1} \left|\frac{a_\nu}{a_n x^{n-\nu}}\right| .$$

Take $|x|$ so large that

$$\frac{1}{|x|} \operatorname{Max} \left\{ \left|\frac{a_\nu}{a_n}\right| \right\} < \frac{1}{2n} ;$$

then

$$\frac{1}{2}|a_n x^n| \leq |P(x)| \leq \frac{3}{2}|a_n x^n| ,$$

and, by the same method, so large that

$$\frac{1}{2}|b_m x^m| \leq |Q(x)| \leq \frac{3}{2}|b_m x^m| .$$

For $m > n + 1$,

$$R(x) \leq 3\left|\frac{a_n x^n}{b_m x^m}\right| \leq \frac{3\left|\frac{a_n}{b_m}\right|}{|x|^{m-n}}$$

and I converges.

For $m \leq n + 1$,

$$R(x) \geq \frac{1}{3} \frac{\left|\frac{a_n}{b_m}\right|}{|x|^{m-n}}$$

and I diverges.

11. Test for convergence

(a) $\displaystyle\int_1^e \frac{dx}{x^m \log x}$.

To examine the behavior of $\log x$ for $x > 1$ apply the Law of the Mean on $[1,x]$ to obtain for some ξ in $(1,x)$,

$$\log x = \log x - \log 1 = \frac{1}{\xi}(x-1) < x - 1 .$$

Thus, if $m \geq 0$

$$f(x) \geq \frac{1}{e^m(x-1)}$$

and, if $m \leq 0$

$$f(x) \geq \frac{1}{(x-1)} .$$

In either case I diverges.

(b) $\displaystyle\int_1^e \frac{dx}{x(\log x)^m}$

$$\int \frac{dx}{x(\log x)^m} = \begin{cases} \dfrac{1}{1-m}\,\dfrac{1}{(\log x)^{m-1}} , & \text{if } m \neq 1 ; \\[2ex] \log\log x , & \text{if } m = 1 . \end{cases}$$

I converges only if $m < 1$.

(c) $\displaystyle\int_e^\infty \frac{dx}{x^m \log x}$

If $m > 1$, $f(x) \leq \frac{1}{x^m}$, I convergent.

If $m = 1$, $\displaystyle\int \frac{dx}{x \log x} = \log\log x$. I divergent.

If $m < 1$, then for $x > e$

$$\frac{1}{x^m \log x} \geq \frac{1}{x \log x} \geq 0$$

and I diverges.

(d) $\displaystyle\int_e^{\infty} \frac{dx}{x(\log x)^m}$

From the indefinite integral given in Part (b), I converges only if $m > 1$.

12. Using estimates in the manner of Example 10-6i, obtain an upper estimate of

$$\int_{\infty}^{\infty} e^{-x^2/2}\, dx \, .$$

First note that

$$I = \int_{\infty}^{\infty} e^{-x^2/2}\, dx = 2\int_0^{\infty} e^{-x^2/2} dx \, .$$

For the integral over $[0,1)$ we use the estimate $e^{-x^2/2} \leq 1$ and for the integral over $[1,\infty]$, $e^{-x^2/2} \leq e^{-x/2}$. Thus, from the result of Number 11(a),

$$I \leq 2\left[\int_0^1 dx + \int_1^{\infty} e^{-x/2}\, dx\right]$$

$$\leq 2 + \frac{4}{\sqrt{e}} < 2 + \frac{4}{1.6} \leq 4.5 \, .$$

(Compare $I = \sqrt{2\pi} \approx 2.51$).

13. Complete the demonstration of Example 10-6j that $\displaystyle\int_0^{\pi/2} \log \sin x \, dx$ converges by proving

$$\sin x \geq \frac{2x}{\pi} \, , \qquad (0 \leq x \leq \tfrac{\pi}{2}) \, .$$

(Hint: sin x is flexed downward on the given interval).

Since sin x is flexed downward on $[0, \frac{\pi}{2}]$ the graph $y = \sin x$ lies above its chord, given by $y = \frac{2x}{\pi}$.

Alternatively, the derivative of $\sin x - \frac{2x}{\pi}$ is $\cos x - \frac{2}{\pi}$ which is decreasing on $[0, \frac{\pi}{2}]$ and changes sign from negative to positive. Thus the zero of the derivative corresponds to an interior maximum of the function. It follows that the function achieves its minimum at an endpoint. Since both endpoint values are zero, the result follows.

14. Give an example of a function f which is Riemann integrable over every closed subinterval $[\alpha,\beta]$ of $(0,1)$ and for which the integrals are bounded,

$$\left|\int_{\alpha}^{\beta} f(x)dx\right| \leq M ,$$

yet which is not itself integrable over $[0,1]$ in the extended sense.

$$f(x) = D_x \sin \frac{1}{x} \quad \text{on} \quad (0,1) \ .$$

15. Prove that if f is continuous on (a,b) and if $\int_a^b f(x)dx$ exists.

Since f is Riemann integrable over every closed subinterval of (a,b), f satisfies the convergence criterion of Theorem 10-6a with $g = f$.

16. Show that if the generalized integral defined by (1) exists, then it is independent of the choice of ξ .

Let ξ and η be any two points of (a,b) . Then

$$\begin{aligned} I &= \lim_{\alpha\sim a^+}\int_{\alpha}^{\xi} f(t)dt + \lim_{\beta\sim b^-}\int_{\xi}^{\beta} f(t)dt \\ &= \lim_{\alpha\sim a^+}\left[\int_{\alpha}^{\eta} f(t)dt + \int_{\eta}^{\xi} f(t)dt\right] + \lim_{\beta\sim b^-}\left[\int_{\eta}^{\beta} f(t)dt + \int_{\xi}^{\eta} f(t)dt\right] . \end{aligned}$$

Apply the limit theorems 3-4c and 3-4a and Definition 6-4b to obtain

$$I = \lim_{\alpha\sim a^+}\int_{\alpha}^{\eta} f(t)dt + \lim_{\beta\sim b^-}\int_{\eta}^{\beta} f(t)dt \ .$$

17. Verify that the value of the integral in the extended sense is not affected if additional points are included in the partition used in Definition 10-6.

Let $u_1 , u_2 , \dots , u_{k-1}$ be additional partition points inserted between x_{i-1} and x_i , set $u_0 = x_{i-1}$, $u_k = x_i$, let ξ be any point between u_0 and u_1 , and let η be any point between u_{k-1} and u_k . Note that

$$\int_a^b f(t)dt = \lim_{\alpha \to u_0^+}\int_\alpha^\xi f(t)dt + \int_\xi^{u_1} f(t)dt$$

$$+ \sum_{j=2}^{k-1} \int_{u_{j-1}}^{u_j} f(t)dt + \int_{u_{k-1}}^{\eta} f(t)dt$$

$$+ \lim_{\beta \to u_k^-} \int_\eta^\beta f(t)dt \ .$$

(From (2), (1) and No. 1)

$$= \lim_{\alpha \to a^+}\int_\alpha^\xi f(t)dt + \int_\xi^\eta f(t)dt$$

$$+ \lim_{\beta \to b^-} \int_\eta^\beta f(t)dt$$

(Theorem 6-4b)

$$= \lim_{\alpha \to u_0^+}\int_\alpha^\xi f(t)dt + \lim_{\beta \to u_k} \int_\xi^\beta f(t)dt$$

(As in No. 16) .

But this last is just the statement of (1).

18. A function f is said to be piecewise continuous on $[a,b]$ if there is a partition $\{x_0,x_1,x_2,\ldots,x_n\}$ of the interval such that f is continuous on each open subinterval (x_{i-1},x_i) and the one-sided limits, $\lim_{x \to x_{i-1}} f(x)$, $\lim_{x \to x_i^+} f(x)$, exist $(i = 1,2,3,\ldots,n)$. Show that f is integrable in the extended sense.

Use Number 1(b).

TC10-7. Linear Differential Equations of First Order.

The place of this section in the text and its importance for applications have been suggested as early as Section 8-5 and throughout Chapter 9.

The operator L transforms functions into functions; i.e., L transforms f into the function $L[f]$. Thus, for example, if the differential operator L is given by

$$L = D^n + a_1 D^{n-1} + \dots + a_n$$

then

$$L[f] = D^n f + a_1 D^{n-1} f + \dots + a_n f$$

where D^n denotes the n^{th} derivative of f.

The derivation of (5) only proves local uniqueness if $y_0 \neq 0$; i.e., that (5) is the only possible solution in some unspecified neighborhood of x_0. The proof of Theorem 10-9a covers the case $y_0 = 0$ and is a global result for the entire domain of p.

Note that the integral of Example 10-7c can be obtained directly from the computation in Example 10-4e, as follows

$$\begin{aligned}\int e^{at} \cos \omega t \, dt &= \frac{e^{at}}{a} \cos \omega t + \frac{\omega}{a} \int e^{at} \sin \omega t \, dt \\ &= \frac{e^{at}}{a} \cos \omega t + \frac{\omega e^{at}}{a(a^2 + \omega^2)} (a \sin \omega t - \omega \cos \omega t) .\end{aligned}$$

Solutions Exercises 10-7

1. Verify the following properties of a linear operator L,

$$L[au] = aL[u]$$
$$L[u + v] = L[u] + L[v]$$

for all functions u and v in the domain of L. Conversely, show if these properties are satisfied then L is a linear operator.

If u and v are in the domain of L, then

$$\begin{aligned}L[au] = L[au + 0 \cdot u] &= aL[u] + 0 \cdot L[u] \\ &= aL[u]\end{aligned}$$

and

$$\begin{aligned}L[u + v] &= L[1 \cdot u + 1 \cdot v] = 1 \cdot L[u] + 1 \cdot L[v] \\ &= L[u] + L[v] .\end{aligned}$$

Conversely, if the stated conditions are satisfied then

$$\begin{aligned} aL[u] + bL[v] &= L[au] + L[bv] \\ &= L[au + bv] \,. \end{aligned}$$

Let L be a linear operator.

(a) Show that the difference between any two solutions of the equation $L[u] = f$ is a solution of the homogeneous equation $L[u] = 0$. Show also that if u is a solution of $L[u] = f$ and v is a solution of the corresponding reduced equation, that $u + v$ is a solution of the original equation.

Suppose $L[u] = L[v] = f$; then

$$L[u - v] = L[u] - L[v] = f - f = 0 \,.$$

Since $L[u] = f$ and $L[v] = 0$, $L[u + v] = L[u] + L[v] = f$.

(b) Verify that any linear combination of solutions of the homogeneous equation $L[u] = 0$, is again a solution.

If $L[u] = L[v] = 0$, then

$$L[au + bv] = aL[u] + bL[v] = a \cdot 0 + b \cdot 0 = 0 \,.$$

(c) Show if $c \neq 0$ that (2) cannot be put in the form of a linear equation

$$L[u] = f \,.$$

Suppose (2) could be written in the form of a linear equation $L[u] = f$. Since 0 is always a solution of the reduced equation, put $u = 0$ in (2) to obtain $a = 0$ for the reduced equation. Thus the reduced equation must have the form

$$Du = bu + cu^2 \,.$$

This equation has the constant solution $u = -\frac{b}{c}$. But if u is a solution of the reduced equation, so is $2u$. Consequently,

$$D[2u] = b(2u) + c(4u^2) \quad \text{or} \quad 0 = \frac{-2b^2}{c} + \frac{4b^2}{c} \,,$$

whence $b = 0$. The reduced equation must therefore have the form

$$Du = cu^2 \,.$$

However $u = -\frac{1}{cx}$ is a solution of this equation, hence so is $\frac{2}{cx}$. Enter $u = \frac{2}{cx}$ in the equation to obtain $-\frac{2}{cx^2} = \frac{4}{cx^2}$, a contradiction.

3. **For the solution of Equation (4) in the form (5) show that choice of a different end of integration $x_1 \neq x_0$ in the domain of p where $u(x_1) \neq 0$ yields the same value of y .**

Replace x_0 by x_1 , y_0 by y_1 in (5) to obtain the solution

$$y = y_1 \exp\left\{-\int_{x_1}^{x} p(t)dt\right\}$$

$$= y_1 \exp\left\{\int_{x_0}^{x_1} p(t)dt - \int_{x_0}^{x} p(t)dt\right\} .$$

The integral from x_0 to x_1 is a constant, k .

For $c = e^{k}$,

$$y = cy_1 \exp\left\{-\int_{x_0}^{x} p(t)dt\right\} .$$

Take $x = x_0$ **to obtain** $y_0 = cy_1$.

4. Verify that the function z defined by (12) is a **particular solution** of (3).

Substitute z for u in $L[u]$.

5. Give the general solutions of the following equations and solve the indicated initial value problem. In what domains are the solutions valid?

(a) $y' + y = 4$; $y = 0$ at $x = 0$.

The equation clearly has the particular solution $y = 4$.

General solution: $y = ce^{-x} + 4$.

The solution of the initial value problem is

$$y = 4 - 4e^{-x} .$$

(b) $y' = ay + b$, $(a,b$ constants); $y = -\frac{b}{a}$ at $x = 0$, $(a \neq 0)$.

Note that the equation has the constant particular solution $y = -\frac{b}{a}$. By the uniqueness theorem this is the solution of the initial value problem.

General solution: $y = ce^{ax} - \frac{b}{a}$.

(c) $xy' - y = x^2$; $y = 4$ at $x = 2$.

Put the equation in the form of (3): $y' - \frac{y}{x} = x$, $(x \neq 0)$. The reduced equation has the general solution

$$v(x) = \exp(-\int \frac{1}{x} dx) = Cx .$$

Apply Equation (12) to obtain the particular solution of the inhomogeneous equation,

$$z(x) = x^2 .$$

General solution: $y = Cx + x^2$.

Although the method of solution excluded the point $x = 0$, this is a valid solution for all x . The solution of the initial value problem is $y = x^2$.

(d) $2e^{-x}y' + e^{x}y = 2e^{x}$; $y = e + 2$ at $x = 0$.

Put the equation in the form of (3): $y' + \frac{1}{2}e^{2x}y = e^{2x}$. This has the constant particular solution $z : x \longrightarrow 2$. The general solution of the reduced equation is $v : x \longrightarrow e^{-e^{2x}/4}$. This yields for the general solution of the nonhomogeneous equation

$$y = Ce^{-e^{2x}/4} + 2 ,$$

and for the solution of the initial value problem, $C = e^{5/4}$.

6. (a) Obtain the general solution, in terms of elementary functions, of the equation

$$y' + xy = x \ .$$

For the fundamental solution take

$$v(x) = \exp\left\{-\int_0^x t \, dt\right\}$$

$$= e^{-x^2/2} \ .$$

The particular solution may be obtained in the usual way, or it may be discovered at once that $z = 1$ is a particular solution.

General solution:

$$y = 1 + ce^{-x^2/2} \ .$$

(b) Contrast with the general solution of

$$y' + xy = 1 \ .$$

The fundamental solution is the same as in Part (a) but now

$$\int_0^x \frac{f(t)}{v(t)} \, dt = \int_0^x e^{t^2/2} dt$$

cannot be expressed in terms of elementary functions.

General solution:

$$y = e^{-x^2/2} [c + \int_0^x e^{t^2/2} \, dt] \ .$$

7. Let u and v satisfy

$$u' + p \cdot u = f \ , \ v' + p \cdot v = g$$

where $f(x) > g(x)$ for $x > x_0$. Show that if $u(x_1) > v(x_1)$ for some $x_1 \geq x_0$ then $u(x) > v(x)$ for all $x > x_1$.

The function $z = u - v$ satisfies the differential equation

$$z' + p \cdot z = h$$

where $h(x) = f(x) - g(x) > 0$ for $x \geq x_1$ and $z(x_1) > 0$. From

$$z(x) = \phi(x)[z(x_1) + \int_{x_1}^{x} \frac{h(t)}{\phi(t)}\,dt]$$

where $\phi(x) = \exp\left\{-\int_{x_1}^{x} p(t)dt\right\} > 0$ it follows that $z(x) > 0$ for $x > x_1$.

8. The solution of Equation (3) was obtained under the assumption that p and f are continuous functions. On the other hand, Formula (13) is meaningful when p and f may be only piecewise continuous (defined in Exercises 10-6b, No. 18). Revise the theory so that it applies more generally to piecewise continuous functions. (Hint: you will have to give up the requirement that solutions are differentiable at every point, but while relaxing this requirement, consider only solutions which are continuous.)

Observe first that p and f are bounded on any bounded interval, hence the functions P and F defined by the integrals

$$P(x) = \int_{x_0}^{x} p(t)dt\,, \quad F(x) = \int_{x_0}^{x} f(t)dt$$

are continuous. For example, if $|P(t)| < M$, then

$$|P(x + h) - P(x)| = |\int_{x}^{x+h} p(t)dt|$$
$$< M|h| < \epsilon\,,$$

provided that $|h| < \frac{\epsilon}{M}$. Thus, so long as p and f are piecewise continuous (or merely integrable), Formula (13) defines a continuous function.

At a point ξ of discontinuity of p or f the right- and left-sided derivatives of $u(x)$ exist since P and F have right- and left-sided derivatives. Thus $u'(\xi^+) = -p(\xi^+)u(\xi) + f(\xi^+)$ where by $f(\xi^+)$ is meant $\lim_{x \to \xi^+} f(x)$, etc., and because of the right-hand continuity of p and f, $u'(\xi^+)$ is the right-hand derivative of u:

$$u'(\xi^+) = \lim_{x \to \xi^+} u'(x) = \lim_{x \to \xi^+} \frac{u(x) - u(\xi)}{x - \xi}\,.$$

9. In the light of Number 8, determine the current in the circuit of Example 10-7c when the electromotive force is a "square wave" of period 2λ :

$$E(t) = \begin{cases} E_0 \text{ , for } 2n\lambda \leq t < (2n+1)\lambda \text{ ,} \\ -E_0 \text{ , for } (2n+1)\lambda \leq t < (2n+2)\lambda \text{ ,} \end{cases}$$

$n = 0,1,2,\ldots$. What is the asymptotic solution as t approaches infinity?

As before the fundamental solution is $v(t) = e^{-at}$. Set $b = \frac{E_0}{L}$. The solution of the initial value problem is then

$$\text{(i)} \qquad u(t) = e^{-at}[I_0 + b \int_0^t e^{as}g(s)ds]$$

where

$$g(s) = \frac{E(s)}{E_0} = (-1)^{[\![s/\lambda]\!]}$$

The initial value I_0 contributes only to the transient term and the problem is to obtain an asymptotic representation of the integral. For this purpose set $t = n\lambda + \tau$ where $n = [\![\frac{t}{\lambda}]\!]$ and $0 \leq \tau < \lambda$. Next, split the integral into intervals of length λ with a remainder term:

$$\int_0^t e^{as}g(s)ds = S(t) + R(t)$$

where

$$S(t) = \sum_{k=0}^{n-1} \int_{k\lambda}^{(k+1)\lambda} e^{as}g(s)ds$$

and

$$R(t) = \int_{n\lambda}^t e^{as}g(s)ds \text{ .}$$

Then

$$S(t) = \sum_{k=0}^{n-1} \int_{k\lambda}^{(k+1)\lambda} (-1)^k e^{as}ds$$

$$= \frac{1}{a} \sum_{k=0}^{n-1} (-1)^k [e^{(k+1)\lambda a} - e^{k\lambda a}]$$

$$= \frac{e^{\lambda a} - 1}{a} \sum_{k=0}^{n-1} (-1)^k e^{k\lambda a} \text{ .}$$

This last sum is a geometrical progression with common ration $-e^{\lambda a}$, hence,

$$\text{(ii)} \qquad S(t) = \frac{e^{\lambda a} - 1}{a} \, \frac{1 - (-1)^n e^{n\lambda a}}{1 + e^{\lambda a}} \; .$$

The remainder term is

$$\text{(iii)} \qquad R(t) = \int_{n\lambda}^{n\lambda + \tau} (-1)^n \, e^{as} ds$$

$$= (-1)^n \frac{e^{n\lambda a}}{a} (e^{a\tau} - 1) \; .$$

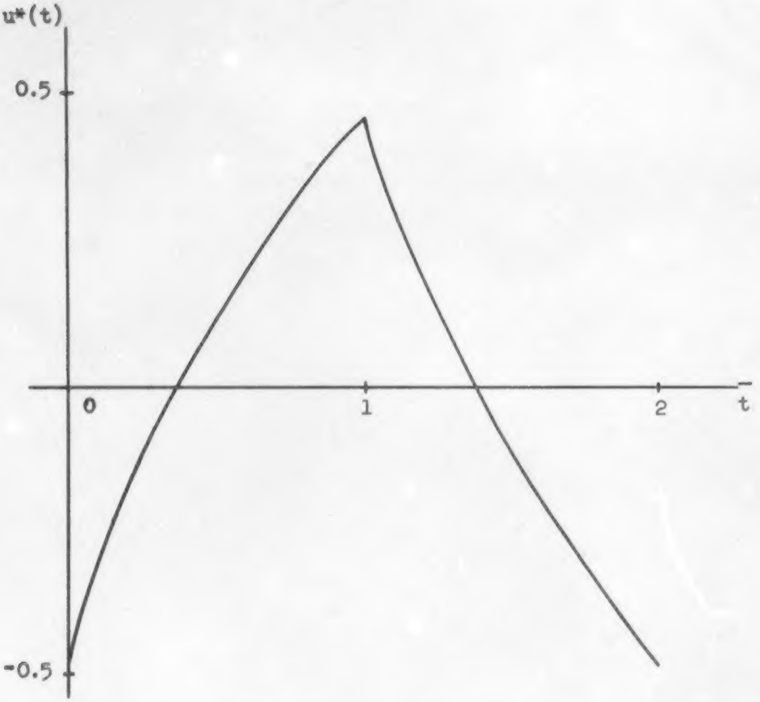
Insert the results of (ii) and (iii) into (i), with $e^{-at} = e^{-n\lambda a} e^{-a\tau}$, to obtain

$$u(t) = (-1)^n \frac{b}{a} e^{-a\tau} \left[\frac{1 - e^{\lambda a}}{1 + e^{\lambda a}} + (e^{a\tau} - 1) \right] + ke^{-\lambda t}$$

where k is a constant. Drop the transient term and collect terms to obtain the asymptotic solution

$$u^*(t) = (-1)^n \frac{b}{a} \left[1 - \frac{2e^{a(\lambda - \tau)}}{e^{\lambda a} + 1} \right] .$$

From this expression it is clear that $u^*(t + 2\lambda) = u^*([n + 2]\lambda + \tau) = u^*(n\lambda + \tau) = u^*(t)$, or that u^* is periodic with period 2λ. The graph of u^* for one cycle is given below, where for convenience $a = b = \lambda = 1$. Note the corners in the graph at $t = 1$ and $t = 2$ corresponding to the discontinuities in the forcing term.



10. Let $p(x) = \frac{a}{x^r}$ in the homogeneous equation (4). Discuss the possibility of finding a solution $y = u(x)$ for $x > 0$ such that $\lim_{x\sim 0^+} u(x)$ exists. What implications does this have for the initial value problem at $x = 0$?

The general solution of the homogeneous equation is either

(a) $u(x) = c \exp\left(\frac{ax^{1-r}}{1 - r}\right)$, for $r \neq 1$;

or

(b) $u(x) = c \exp(a \log x) = cx^a$, for $r = 1$.

In (a), if $r < 1$, $\lim_{x\sim 0^+} u(x) = c$ and the initial value problem at $x = 0$ is well posed. If $r > 1$, the limit exists only if $a \geq 0$ and in that case $\lim_{x \sim 0^+} u(x) = 0$; thus there is no freedom to assign the initial value at $x = 0$ and the initial value problem at that point is not well-posed (a solution fails to exist if $y_0 \neq 0$) . In (b), if $a > 0$, $\lim_{x \sim 0^+} u(x) = 0$ and the initial value problem is not well-posed; if $a = 0$, $\lim_{x \sim 0^+} u(x) = c$ and the initial value problem is well-posed; if $a < 0$, no limit exists.

11. In the text it was stated that the solution depends continuously on the initial value y_0 and the functions p and f . The idea of continuous dependence on the initial values is clear: for a given value of x , y is a continuous (in this case, linear) function of y_0 . But what can be meant by continuous dependence on p and f ? Give your interpretation. Then verify that (14) does staisfy the continuous dependence conditions.

There are many possible answers to this question. The following is a natural development from the earlier ideas of the text. It is unlikely that a student will invent a more sophisticated measure of magnitude for a function f than $\text{lub}|f(x)|$.

Take $x > x_0$. Since $u(x)$ depends on all values of $p(t)$ for t in $[x_0,x]$ the definition of continuity must be modified accordingly. Let $p(x)$ be replaced by $p(x) + d(x)$ in (3) and let the corresponding solution be written $u(x) + e(x)$. The solution u is said to be continuously dependent on p if for each $\epsilon > 0$ it is possible to guarantee $|e(x)| < \epsilon$ by requiring $|d(t)| < \delta$ for some positive δ and all t in $[x_0,x]$. The same applies to f .

The proof of continuous dependence is simple in conception and straightforward in execution, although it may involve many details. The proof for p (the longer one) is given here.

In (14), p affects the solution only insofar as it affects the fundamental solution v . Let the fundamental solution for $p(x) + d(x)$ be denoted by $v(x) + \omega(x)$. Then

$$\omega(x) = \exp\left\{-\int_{x_0}^{x} [p(t) + d(t)]dt\right\} - \exp\left\{-\int_{x_0}^{x} p(t)dt\right\}$$

$$= \exp\left\{-\int_{x_0}^{x} p(t)dt\right\}$$

$$\cdot \left[\exp\left\{-\int_{x_0}^{x} d(t)dt\right\} - 1\right] .$$

Set $I = -\int_{x_0}^{t} d(s)ds$. Then, for the expression in brackets, we have by the Law of the Mean,

$$e^I - 1 = e^I - e^0 = e^{\xi} I$$

where ξ is some number between 0 and I . Now $|I| < \int_{x_0}^{t} |d(s)|ds < \delta(t - x_0)$, and if M is an upper bound on $[x_0,x]$ for $v(t) = \exp\left\{-\int_{x_0}^{t} p(t)dt\right\}$, then

(i) $$|\omega(t)| < \delta(t - x_0)M < \delta(x - x_0)M .$$

Now, compare $u(x) + e(x)$ with $u(x)$:

(ii) $$e(x) = \omega(x)\left[y_0 + \int_{x_0}^{x} \frac{f(t)}{v(t) + \omega(t)} dt\right]$$

$$+ vx\int_{x_0}^{x} f(t)\left\{\frac{1}{v(t) + \omega(t)} - \frac{1}{v(t)}\right\}dt .$$

Now we show that each term in the representation of $e(x)$ can be made arbitrarily small by choosing δ sufficiently small. Let m be the minimum of the continuous function v on $[x_0,x]$; since v is positive, $m > 0$. Let N be an upper bound for $|f(t)|$. Then

(iii) $$\left|\frac{f(t)}{v(t) + \omega(t)}\right| \leq \frac{|f(t)|}{|v(t)| - |\omega(t)|} < \frac{2N}{m}$$

where we have chosen δ in (i) so that $|v(t)| - |\omega(t)| > \frac{m}{2}$, namely,

(iv) $$\delta < \frac{m}{2(x - x_0)M} .$$

Furthermore,

(v)
$$\left| f(t) \left\{ \frac{1}{v(t) + \omega(t)} - \frac{1}{v(t)} \right\} \right|$$
$$\leq \frac{|f(t)|\ |\omega(t)|}{v(t)[v(t) - |\omega(t)|]}$$
$$< \frac{\delta(x - x_0)MN}{m^2/2} .$$

Enter the results of (i), (iii) and (v) in (ii) to obtain

$$|\epsilon(x)| < \delta \left\{ |y_0|(x - x_0)M + \frac{2(x - x_0)^2 MN}{m} + \frac{2(x - x_0)M^2 N}{m^2} \right\} .$$

Condition (iv) must be satisfied, but is clear that the error tolerance ϵ can be met by taking δ small enough.

TC10-8. Linear Differential Equations of Second Order.

In this section it is important to stress the solution of second order equations with constant coefficients. This will be used later in the text, particularly in Chapter 12. The general theory is extremely important for higher applications, but will not have as important a role here.

We are still employing techniques of integration to solve these differential equations and we avoid problems that cannot be solved in these terms.

Solutions Exercises 10-8a

1. Verify that the operator

$$L = D^2 + p \cdot D + q$$

is linear.

Use the linearity of D, (Theorem 4-2a). Note, in particular, that

$$D^2(au + bv) = D[D(au + bv)] = D[aDu + bDv]$$
$$= aD^2u + bD^2v \; .$$

2. Prove the superposition principle for linear operators.

The principle is simply a statement of linearity: if $L[u_1] = f_1$ and $L[u_2] = f_2$, then

$$L[c_1u_1 + c_2u_2] = c_1L[u_1] + c_2L[u_2] = c_1f_1 + c_2f_2 \; .$$

3. Prove that the reduced equation $L[u] = 0$, L linear, always has the trivial solution $u : x \longrightarrow 0$.

Take any v in the domain of L . Then

$$L[0] = L[0 \cdot v + 0 \cdot v] = 0 \cdot L[v] + 0 \cdot L[v] = 0 \; .$$

4. If L and M are linear operators, show that the composition LM is linear.

$$LM[au + bv] = L\Big[aM[u] + bM[v]\Big] = a\,LM[u] + b\,LM[v] \; .$$

5. Take $L = (D - \alpha)(D - \beta)$.

(a) Solve the homogeneous equation $L[u] = 0$ by the method indicated in (5).

From Section 10-7 the general solution of the system (5) is given by $v(x) = k_1 e^{\alpha k}$, $u(x) = e^{\beta x}[k_2 + \int_{x_0}^{x} \frac{v(t)}{e^{\beta t}} dt]$. For the solution u of the second order equation, specifically,

$$u(x) = k_2 e^{\beta x} + k_1 e^{\beta x} \int_{x_0}^{x} e^{(\alpha-\beta)t} dt .$$

If $\alpha \neq \beta$, then

$$u(x) = c_1 e^{\alpha x} + c_2 e^{\beta x}$$

where $c_1 = \frac{k_1}{\alpha - \beta}$ and $c_2 = k_2 - \frac{k_1 e^{(\alpha-\beta)x_0}}{\alpha - \beta}$.

If $\alpha = \beta$, then

$$u(x) = e^{\alpha x}(c_1 + c_2 x) ,$$

where $c_1 = k_2 - k_1 x_0$, and $c_2 = k_1$.

(b) Show that the initial value problem $L[u] = 0$ with the prescribed initial data, $u(x_0) = y_0$, $u'(x_0) = y_0'$, is well-posed.

Since any solution of $L[u] = 0$ is a solution of the first order system (5), and viceversa, the existence theory of the second order equation (7) is reduced to the existence theory for linear first order equations. From Section 10-7, the initial value problem for (5) with initial data $u(x_0) = y_0$ and $v(x_0) = z_0$ is well posed. Take $x = x_0$ in (5) to obtain

$$u'(x_0) - \beta u(x_0) = v(x_0) .$$

Thus the initial value problem for the second order equation with initial data $u(x_0) = y_0$, $u'(x_0) = y_0'$ has the solution u of the first order system with initial data $u(x_0) = y_0$, $v(x_0) = y_0' - \beta y_0$.

Furthermore, since the solution of the first order system is unique, so also is that of the second order equation.

6. Show that the differential operators $D - p$ and $D - q$ commute if and only if p and q differ by a constant.

If the operators commute then

$$(D - p)(D - q) = D^2 - (p + q) \cdot D + p \cdot q - q'$$

$$= (D - q)(D - p) = D^2 - (p + q) \cdot D + p \cdot q - p' \ .$$

Consequently, $p' = q'$ and $p = q + C$.

Conversely, if $p = q + C$, then $p' = q'$ and the expansions of the two products above are equal.

7. Find the general solutions of the following differential equations and obtained solutions of the stated initial value problems.

(a) $D^2 u = 0$; $u(0) = 1776$, $u'(0) = 1929$.

General solution: $u = ax + b$. Solution of initial value problem:

$$u = 1929x + 1776 \ .$$

(b) $(D^2 + 2D + 2)u$; $u(0) = \frac{\sqrt{2}}{2}$, $u'(0) = 0$.

General solution: $e^{-x}(a \cos x + b \sin x)$. Solution of initial value problem:

$$\frac{\sqrt{2}}{2} e^{-x}(\cos x + \sin x) = e^{-x} \sin (x + \frac{\pi}{4}) \ .$$

(c) $(D^2 + 2D - 2)u = 0$; $u(0) = 0$, $u'(0) = \sqrt{3}$.

General solution: $e^{-x}(ae^{\sqrt{3}x} + be^{-\sqrt{3}x})$. Solution of initial value problem:

$$\frac{1}{2} e^{-x}(e^{\sqrt{3}x} - e^{-\sqrt{3}x}) = e^{-x} \sinh \sqrt{3}\, x \ .$$

(d) $(D^2 + D - 2)u = 0$; $u(0) = 1$, $u'(0) = 2$.

General solution: $ae^{x} + be^{-2x}$. Solution of initial value problem:

$$\frac{1}{3}(4e^{x} - e^{-2x}) \ .$$

8. In analogy with (12) express the solution of (7) in terms of hyperbolic functions.

Rewrite $L = (D - \alpha)(D - \beta)$ in a form analogous to (11):

$$L = (D + a)^2 - \omega^2$$

where $a = -\frac{\alpha + \beta}{2}$ and $\omega^2 = a^2 - \alpha\beta = \frac{(\alpha - \beta)^2}{4}$.

Take $\omega = \frac{\alpha - \beta}{2}$. Then $\alpha = \omega - a$, $\beta = -\omega - a$. Insert the expressions for α and β in the solution (8):

$$\begin{aligned} u(x) &= c_1 e^{\alpha x} + c_2 e^{\beta x} \\ &= c_1 e^{(\omega - a)x} + c_2 e^{-(\omega + a)x} \\ &= e^{-ax}(c_1 e^{\omega x} + c_2 e^{-\omega x}) \\ &= e^{-ax}[c_1(\cosh \omega x + \sinh \omega x) + c_2(\cosh \omega x - \sinh \omega x)] \\ &= e^{-ax}(k_1 \cosh \omega x + k_2 \sinh \omega x) \end{aligned}$$

where $k_1 = c_1 + c_2$ and $k_2 = c_1 - c_2$.

9. Discuss the graph of the damped sinusoidal function u of (12) for $a = 1$, $\omega = 1$, $u(0) = 1$, $u'(0) = 1$. Pay particular attention to zeros, extrema, points of inflection and points where the graph meets the curves $y = \pm e^{-x}$.

Use the form (12b),

$$u(x) = Ae^{-x} \cos(x - \phi) .$$

Then $u(0) = A \cos \phi = 1$. Take the derivative

$$\begin{aligned} u'(x) &= -Ae^{-x}[\cos(x - \phi) + \sin(x - \phi)] , \\ &= -\sqrt{2}\, Ae^{-x} \cos(x - \phi - \tfrac{\pi}{4}) \end{aligned}$$

and obtain

$$u'(0) = -A[\cos \phi - \sin \phi] = -1 .$$

Use $A \cos \phi = 1$ in this equation to obtain $A \sin \phi = 0$. It follows that $A = 1$ and $\phi = 0$ (we may freely impose $0 \leq \phi < 2\pi$) .

$$u(x) = e^{-x} \cos x$$

$$u'(x) = -\sqrt{2}\, e^{-x} \cos(x - \tfrac{\pi}{4})$$

$$u''(x) = 2e^{-x} \cos(x - \tfrac{\pi}{2}) = 2e^{-x} \sin x \; .$$

The zeros of u occur at $(n + \frac{1}{2})\pi$. The zeros of u' occur at $(n - \frac{1}{4})\pi$ corresponding to maxima at $x = 2n\pi - \frac{\pi}{4}$ and minima at $x = (2n + 1)\pi - \frac{\pi}{4}$. The inflection points occur at $x = n\pi$. The graph touches (i.e., has a common tangent with) the graph of e^{-x} when $x = 2n\pi$, the graph of $-e^{-x}$ when $x = (2n + 1)\pi$. Thus the extrema precede the contact points by the constant amount $\frac{\pi}{4}$. The inflection points coincide with the contact points. These special relations do not hold for other values of a and ω .

10. (a) Show if the roots of the characteristic equation (6) are complex then all solutions of the homogeneous equation (3a) have graphs obtained from some nontrivial solution by a change of scale along the y-axis, $(x,y) \longrightarrow (x,Cy)$, and a translation in the direction of the x-axis, $(x,y) \longrightarrow (x + \xi , y)$.

The indicated geometrical transformations applied to the graph $y = u(x)$, yields the two parameter family

$$\text{(a)} \qquad y = Cu(x - \xi) \; .$$

If $u(x)$ is a solution of (3a) then so is $Cu(x - \xi)$. In the case of two complex roots, it follows from the form (12b) of the solution that any solution can be written in the form (a), where $u(x) = e^{-ax} \cos \omega x$, $C = Ae^{a\xi}$ and $\xi = \frac{\phi}{\omega}$.

(b) Investigate the other two classes of solutions of (3a) with respect to this geometrical property.

When the roots of the characteristic equation are equal, there are two cases. If $c_2 \neq 0$ the general solution (10) can be written in the form (a) with $u = e^{-ax} x$, $\xi = \frac{c_1}{c_2}$ and $C = c_2 e^{-a\xi}$ and may take $C = c_1$, $\xi = 0$. We have, then two geometrically distinct cases (not counting the trivial solution which is always present). For if $e^{-ax} \neq Ce^{-a(x-\xi)}(x - \xi)$ then $C(x - \xi) = e^{a\xi}$; i.e., since C cannot then be zero, $x - \xi$ must be constant for all x .

When the roots of the characteristic equation are real and distinct use the form of the solution in Number 8:

$$y = e^{-ax}(k_1 \cosh \omega x + k_2 \sinh \omega x) \ .$$

Here there are four geometrically distinct cases. If $k_1^2 > k_2^2$ define ξ by $\cosh \omega\xi = \frac{|k_1|}{A}$, $\sinh \omega\xi = -\frac{k_2 \operatorname{sgn} k_1}{A}$ where $A = \sqrt{k_1^2 - k_2^2}$. Then

$$y = (e^{a\xi} \operatorname{sgn} k_1)e^{-a(x-\xi)} \cosh \omega(x - \xi) \ .$$

Thus, take $u(x) = e^{-ax} \cosh \omega x$ and $C = e^{a\xi} \operatorname{sgn} k_1$. If $k_1^2 < k_2^2$ define ξ by

$$\cosh \omega\xi = \frac{|k_2|}{A} , \ \sinh \omega\xi = -\frac{k_1 \operatorname{sgn} k_2}{A} \ .$$

Take $u(x) = e^{-ax} \sinh \omega x$ and $C = e^{a\xi} \operatorname{sgn} k_2$. Finally, if $k_1^2 = k_2^2$ there are the two cases $k_1 = \pm k_2$ and we obtain corresponding one parameter families of solutions

$$y = k_1 e^{(-a \pm \omega)x} \ .$$

In the respective cases, take $u(x) = e^{(-a \pm \omega)x}$, $C = k_1$, $\xi = 0$.

As in Part (b) it is easy to show that the four cases are geometrically distinct by considering $v(x) = u(x)e^{ax}$. In the first case $v(x)$ approaches infinity for both large positive and large negative x ; in the second $v(x)$ approaches infinity for large positive x , minus infinity for large negative x ; in the third and fourth cases $v(x) = e^{\omega x}$ and $v(x) = e^{-\omega x}$, the graphs have the asymptote $x = 0$ on the left and right, respectively and also approach infinity for opposite signs of x .

Solutions Exercises 10-8b

1. Show for each of the special cases (7), (8), (9) of Equation (3) that there exist solutions ϕ and ψ satisfying the initial conditions

$$\begin{cases} \phi(0) = 1 \\ \phi'(0) = 0 , \end{cases} \qquad \begin{cases} \psi(0) = 0 \\ \psi'(0) = 1 , \end{cases}$$

hence that the initial value problem is well-posed.

When the characteristic equation (16) has real distinct roots take the general solution in the form given by Exercises 10-8a, Number 8.

$$y = e^{-ax}(k_1 \cosh \omega x + k_2 \sinh \omega x)$$

then

$$y_0 = k_1 \quad , \quad y_0' = -ak_1 + \omega k_2 \ .$$

Thus, for ϕ take $k_1 = 1$, $k_2 = \frac{a}{\omega}$; for ψ , take $k_1 = 0$, $k_2 = \frac{1}{\omega}$.

When the roots of (6) are equal use the general solution in the form

$$y = e^{-ax}(c_1 + c_2 x) \ .$$

Then $y_0 = c_1$, $y_0' = c_2 - ac_1$. For ϕ take $c_1 = 1$, $c_2 = a$; for ψ , $c_1 = 0$, $c_2 = 1$.

When the roots of (6) are complex use the solution (12b)

$$y = Ae^{-ax} \cos(\omega x - \theta) \ .$$

Then $y_0 = A \cos \theta$, $y' = -A(a \cos \theta - \psi \sin \theta)$. For ϕ , use $A = \frac{1}{\cos \theta}$, $\tan \theta = \frac{a}{\omega}$ to obtain $\theta = \arctan \frac{a}{\omega}$, $A = \frac{1}{\omega}\sqrt{a^2 + \omega^2}$ where we restrict $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Consequently,

$$\phi(x) = e^{-ax}[\cos \omega x + \frac{a}{\omega} \sin \omega x] \ .$$

From $\psi(0) = A \cos \theta = 0$, $\psi'(0) = A \omega \sin \theta = 1$. Take $\theta = \frac{\pi}{2}$, $A = \frac{1}{\omega}$ to obtain $\psi(x) = \frac{1}{\omega} e^{-ax} \sin \omega x$.

2. Find a fundamental set of solutions of

$$L[u(x)] = u''(x) + xu'(x) = 0$$

then solve the initial value problems

(a) $L[u(x)] = 1$; $u(0) = u'(0) = 0$.

(b) $L[u(x)] = x$; $u(0) = u'(0) = 0$.

Set $v = u'$ where u is a solution of the reduced equation ; then v satisfies the first order equation $v'(x) + xv(x) = 0$. Hence, $v(x) = Ae^{-x^2/2}$, from which

$$u(x) = A \int_0^x e^{-t^2/2}\, dt + B \ .$$

A fundamental set $\{\phi,\psi\}$ is given by

$$\phi(x) = \int_0^x e^{-t^2/2}\,dt \ , \ \psi(x) = 1 \ .$$

(a) Set $v = u'$ and obtain a particular solution of the first order equation $v'(x) + xv(x) = 1$; then integrate to obtain a particular solution of $L[u(x)] = 1$. By Section 10-7

$$v(x) = e^{-x^2/2} \int_0^x e^{t^2/2}\,dt \ .$$

The particular solution

$$u(x) = \int_0^x \left\{ e^{-s^2/2} \int_0^s e^{t^2/2}\,dt \right\} ds$$

satisfies the initial conditions.

(b) Proceed as in (a), but, more simply, note that $u' = 1$ is a particular solution of the first order equation for u' . Solution of the initial value problem:

$$u(x) = x - \int_0^x e^{-t^2/2}\,dt \ .$$

3. Verify directly by differentiation that (24) defines a particular solution of (1) and show that the zero initial conditions (25) are satisfied.

Differentiate in (24) to obtain

$$v'(x) = -\phi'(x) \int_{x_0}^x \frac{\psi(t)}{w(t)} f(t)dt + \psi'(x) \int_{x_0}^x \frac{\phi(t)}{w(t)} f(t)dt$$

and

$$v''(x) = -\phi''(x) \int_{x_0}^x \frac{\psi(t)}{w(t)} f(t)dt + \psi''(x) \int_{x_0}^x \frac{\phi(t)}{w(t)} f(t)dt + \frac{\phi(x)\psi'(x) - \psi(x)\phi'(x)}{w(x)} f(t)$$

where the numerator in the last term is the Wronskian $w(x)$. Thus

$$L[v(x)] = f(x) - L[\phi(x)] \int_{x_0}^{x} \frac{\psi(t)}{w(t)} f(t)dt$$

$$+ L[\psi(x)] \int_{x_0}^{x} \frac{\phi(t)}{w(t)} f(t)dt \ .$$

Since $L[\phi] = L[\psi] = 0$, v is a solution of (1). Furthermore, it is immediate from the expressions for v and v' that $v(x_0) = v'(x_0) = 0$.

4. Show how to solve the initial value problem (21) when the lower ends of integration in the expression (25) for the particular solution may be different from x_0 .

Suppose

$$z(x) = -\phi(x) \int_{a}^{x} \frac{\psi(t)}{w(t)} f(t)dt + \psi(x) \int_{b}^{x} \frac{\phi(t)}{w(t)} f(t)dt \ ,$$

where a and b are arbitrary. Then set

$$c_1 = \int_{a}^{x_0} \frac{\psi(t)}{w(t)} f(t)dt \ , \ c_2 = \int_{b}^{x_0} \frac{\phi(t)}{w(t)} f(t)dt \ .$$

The particular solution with zero initial data given by (21) is then $v(x) = z(x) + c_1\phi(x) - c_2\psi(x)$. Apply (24) to get the solution of the initial value problem:

$$u(x) = (y_0 + c_1)\phi(x) + (y_0' - c_2)\psi(x) + z(x) \ .$$

5. Describe the solution of the initial value problem for the general second-order linear equation (1) in operator symbolism by suitably restricting the domain of the differential operator and giving the exact inverse integral operator.

Restrict the domain of the differential operator L of (26) to the set of all twice continuously differential functions u satisfying the conditions $u(x_0) = y_0$, $u'(x_0) = y_0'$. The inverse operator L^{-1} is defined for all continuous functions f by

$$L^{-1}[f] = y_0\phi + y_0'\psi + T[f]$$

where T is the integral operator of (28) and $\{\phi,\psi\}$ is a fundamental set for L at x_0 .

6. Determine the Green's function for the operator with constant coefficients.

For the three cases, take the functions ϕ and ψ from Number 1, and insert in (27). In all three cases, use $w(\xi) = \exp\left\{-\int_0^\xi p(\xi)d\xi\right\} = e^{-2a\xi}$.

When the roots of the characteristic equation (6) are real and distinct,

$$\phi(x) = \frac{e^{-ax}}{\omega}[\omega\cosh\omega x + a\sinh\omega x] ,$$

$$\psi(x) = \frac{e^{-ax}}{\omega}\sinh\omega x .$$

Then, from the addition theorem for sinh ,

$$G(x,\xi) = \frac{1}{\omega}e^{-a(x-\xi)}\sinh\omega(x-\xi) .$$

In exactly the same way, when the roots are complex, then

$$G(x,\xi) = \frac{1}{\omega}e^{-a(x-\xi)}\sin\omega(x-\xi)$$

When the roots are real and equal, $\phi(x) = e^{-ax}(1+ax)$, $\psi(x) = xe^{-ax}$. Then

$$G(x,\xi) = (x-\xi)e^{-a(x-\xi)} .$$

7. The theory of Equation (13a) is based on the assumption that p and q are continuous. This theory can be extended to piecewise continuous functions p and q , if the requirement that solutions be twice continuously differentiable is weakened to require that they be only once continuously differentiable. Assuming the validity of this assertion, construct the fundamental set of solutions at $x = 0$ for the following equations:

(a) $y'' + (\text{sgn } x)y = 0$.

For $x > 0$, the equation has the solutions $\cos x$, $\sin x$, and for $x < 0$, $\cosh x$ and $\sinh x$. Take

$$\phi(x) = \begin{cases} \cos x , & \text{for } x \geq 0 , \\ \cosh x , & \text{for } x \leq 0 , \end{cases}$$

and

$$\psi(x) = \begin{cases} \sin x , & \text{for } x \geq 0 , \\ \sinh x , & \text{for } x \leq 0 . \end{cases}$$

The differential equations and both initial conditions (14a,b) are clearly satisfied.

(b) $y'' + k(x)y = 0$, where

$$k(x) = \begin{cases} 1 , & \text{if } 2n\pi \le x < (2n + 1)\pi , \\ 5 , & \text{if } (2n + 1)\pi \le x < (2n + 2)\pi , \end{cases}$$

$(n = 0 , \pm 1 , \pm 2 , \ldots)$.

In each interval $2n\pi \le x < (2n + 1)\pi$, $\cos x$ and $\sin x$ are solutions. In each interval $(2n + 1)\pi \le x < (2n + 2)\pi$, $\cos \sqrt{5}\, x$ and $\frac{1}{\sqrt{5}} \sin \sqrt{5x}$ are solutions which extend the preceding solutions continuously. Take

$$\phi(x) = \begin{cases} \cos x , & \text{for } 2n\pi \le x \le (2n + 1)\pi , \\ \cos \sqrt{5}\, x , & \text{for } (2n + 1)\pi \le x \le (2n + 2)\pi , \end{cases}$$

and

$$\psi(x) = \begin{cases} \sin x , & \text{for } 2n\pi \le x \le (2n + 1)\pi , \\ \frac{1}{\sqrt{5}} \sin \sqrt{5}\, x , & \text{for } (2n + 1) \le x \le (2n + 2)\pi . \end{cases}$$

8. (a) Construct the asymptotic solution as x approaches infinity for the equation

$$y'' + 4y' + 3 = k(x)$$

where k is defined in Number 7.

The characterictic equation has distinct real roots. Put the differential operator in the standard form

$$(D + 2)^2 - 1^2 .$$

In the solution (26) the first two terms are transient and we need consider only $v(x) = \int_0^x G(x,\xi)k(\xi)d\xi$, where from Number 5

$$G(x,\xi) = e^{-2(x-\xi)} \sinh (x - \xi) = \frac{1}{2} [e^{-(x-\xi)} - e^{-3(x-\xi)}]$$

To simplify the work, set $k(x) = 2g(x) + 3$ where $g(x) = (-1)^{m+1}$ and $m = \left[\!\left[\frac{x}{\pi}\right]\!\right]$ (compare Exercises 10-7, No.9). The exact solution is $2F(x) + 3H(x)$ where $F(x) = \int_0^x G(x,\xi)g(\xi)d\xi$ and

$H(x) = \int_0^x G(x,\xi)d\xi$. Utilize the indefinite integral

$$\int G(x,\xi)d\xi = \frac{1}{6}\,[3e^{-(x-\xi)} - e^{-3(x-\xi)}]\ .$$

Now, set $r = x - m\pi$. Put

$$F(x) = \int_0^x G(x,\xi)g(\xi)d\xi = I(x) + J(x)$$

where

$$I(x) = (-1)^{m+1}\int_{m\pi}^{m\pi+r} G(m\pi + r\,,\xi)d\xi\ ,$$

$$J(x) = \sum_{\mu=0}^{m-1} (-1)^{\mu+1}\int_{\mu\pi}^{(\mu+1)\pi} G(x,\xi)d\xi\ .$$

Then

$$I(x) = \frac{(-1)^{m+1}}{6}\,[2 - (3e^{-r} - e^{-3r})]$$

and

$$J(x) = \sum_{\mu=0}^{m-1} \frac{(-1)^{\mu+1}}{6}\, 3e^{-x}[e^{(\mu+1)\pi} - e^{\mu\pi}]$$

$$- e^{-3x}[e^{3(\lambda+1)\pi} - e^{3\mu\pi}]$$

$$= -\frac{e^{-x}}{2}\,(e^{\pi} - 1)\sum_{\mu=0}^{m-1}(-e^{\pi})^{\mu} + \frac{e^{-3x}}{6}\,(e^{3\pi-1})\sum_{\mu=0}^{m-1}(-e^{3\pi})^{\mu}$$

$$= -\frac{e^{-x}}{2}\,(e^{\pi} - 1)\frac{1 - (-1)^{m}e^{m\pi}}{1 + e^{\pi}}$$

$$+ \frac{e^{-3x}}{6}\,(e^{3\pi} - 1)\,\frac{1 - (-1)^{m}e^{3m\pi}}{1 + e^{3\pi}}\ .$$

Set $m\pi = x - r$ and drop transient terms to obtain the asymptotic expression for $J(x)$:

$$J^{*}(x) = (-1)^{m}[\frac{e^{-r}}{2}\,\frac{e^{\pi} - 1}{e^{\pi} + 1} - \frac{e^{-3r}}{6}\,\frac{e^{3\pi} - 1}{e^{3\pi} + 1}]\ .$$

Add to obtain the asymptotic expression for the integral $F(x)$:

$$F^{*}(x) = I(x) + J^{*}(x) = (-1)^{m+1}[\frac{e^{-r}}{e^{\pi} + 1} - \frac{1}{3}\,\frac{e^{-3r}}{e^{3\pi} + 1} + \frac{1}{3}]\ .$$

To complete the solution, compute

$$H(x) = \int_0^x g(x,\xi)\,d\xi = \frac{1}{2} - \frac{1}{6}(3e^{-x} - e^{-3x})$$

and drop transient terms to obtain

$$H^*(x) = \frac{1}{2} \, .$$

The total asymptotic solution is

$$2F^*(x) + 3H^*(x) = 2(-1)^{m+1}\left[\frac{e^{-r}}{e^{\pi} + 1} - \frac{1}{3}\,\frac{e^{-3r}}{e^{3\pi} + 1} + \frac{1}{3}\right] + \frac{3}{2} \, .$$

(b) Do the same for the equation

$$y'' + 2y' + 2 = k(x) \, .$$

In standard form, the operator is

$$(D + 1)^2 + 1^2 \, .$$

From Number 5 we then have

$$G(x,\xi) = e^{-(x-\xi)} \sin\,(x - \xi) \, .$$

Define g, F, F^*, H, H^*, I, J, J^* as before. From Example 10-4e, obtain the indefinite integral

$$\int G(x,\xi)\,d\xi = \frac{1}{2}\, e^{-(x-\xi)}[\sin(x - \xi) + \cos(x - \xi)] \, .$$

Hence

$$H(x) = \int_0^x G(x,\xi)\,d\xi = \frac{1}{2} - e^{-x}\,[\sin x + \cos x]$$

and $H^*(x) = \frac{1}{2}$. Next

$$I(x) = \frac{(-1)^{m+1}}{2}\,[1 - e^{-r}(\sin r + \cos r)] \, .$$

Finally, use $x = m\pi + r$ and $\cos n\pi = (-1)^n$ to get

$$J(x) = \frac{e^{-x}}{2}\sum_{\mu=0}^{m-1} (-1)^m(e^{(\mu+1)\pi} + e^{\mu\pi})(\sin r + \cos r)$$

$$= \frac{1}{2}(-1)^m e^{-x}(\sin r + \cos r)(e^{\pi} + 1) \sum_{\mu=0}^{m-1} e^{\mu\pi}$$

$$= \frac{1}{2}(-1)^m(\sin r + \cos r)(e^{r} + 1)\left[\frac{e^{m\pi} - 1}{e^{\pi} - 1} e^{-x}\right] .$$

Drop the transient in the final bracket to obtain

$$J^*(x) = \frac{(-1)^m}{2} \frac{e^{r} + 1}{e^{\pi} - 1} e^{-r}(\sin r + \cos r) .$$

Consequently

$$F^*(x) = I(x) + J^*(x)$$

$$= (-1)^m[e^{\pi - r} \sin(r + \frac{\pi}{4}) - \frac{1}{2}] .$$

The total asymptotic solution is then

$$2F^*(x) + 3H^*(x) = (-1)^m[e^{\pi - r} \sin(r + \frac{\pi}{4}) - \frac{1}{2}] + \frac{3}{2} .$$

TC10-9. Separable Differential Equations.

In Theorem 10-9, to show that uniqueness may fail if $g(y_0) = 0$, consider the simple equation and initial condition

$$y' = 2\sqrt{y} \; ; \; y = 0 \, , \text{ at } x = a \, .$$

Observe that $g : y \longrightarrow \sqrt{y}$ is not differentiable at $y = 0$ - a sign of potential nonuniqueness. The differential equation has a family of solutions given by

$$\int \frac{dy}{2\sqrt{y}} = \int dx$$

or

$$\sqrt{y} = x + C \, ,$$

or

$$y = (x + C)^2 \, .$$

These are parabolas tangent to the x-axis. Enter the initial condition to obtain the solution

$$y = (x - a)^2 \, .$$

However, $y = 0$ also satisfies the differential equation and initial condition.

Solutions Exercises 10-9

1. Solve the equation

$$y' + xy = x$$

by separating variables. (Compare Exercises 10-7, No. 6).

Note that,

$$y' = x(1 - y)$$

whence

$$\log |y - 1| = - \frac{x^2}{2} + C \, ,$$

or

$$y = 1 + \exp \left\{ - \frac{x^2}{2} + C \right\} \operatorname{sgn} y \, .$$

The solution $y = 1$ is not of this form, but all solutions can be expressed in the form

$$y = 1 + k \, e^{-x^2/2} \, .$$

2. Find the orthogonal trajectories to each of the following families of curves and sketch the curves and their orthogonal trajectories.

(a) The rectangular hyperbolas, $xy = a$.

For the original family,

$$xy' + y = 0 ,$$

whence

$$y' = -\frac{y}{x} .$$

So, for the orthogonal family

$$y' = \frac{x}{y} .$$

Integrate to obtain the family of hyperbolas given by

$$y^2 - x^2 = C .$$

(b) The ellipses centered at the origin with fixed ratio of major to minor axis.

Let the ellipses be given in the form

$$\frac{x^2}{\lambda} + y^2 = a^2$$

where $\lambda > 0$ is the fixed ratio of the axes and a^2 is the parameter of the problem. Then for the original family

$$y' = -\frac{x}{\lambda y}$$

and for the orthogonal trajectories

$$y' = \frac{\lambda y}{x} .$$

The equation for the orthogonal trajectories is separable and has the solutions $y = 0$ and $\log |y| = \lambda \log |x| + C$. All solutions can be expressed in the form

$$y = k |x|^{\lambda} .$$

3. (a) Show that the equation

$$\frac{dy}{dx} = \phi\left(\frac{y}{x}\right)$$

is transformed into a separable equation in $\frac{dy}{dv}$ by means of the substitution $x = \frac{y}{v}$.

Set $y = xv$. Then $y' = xv' + v$. Consequently, v satisfies the separable equation

$$v' = \frac{\phi(v) - v}{x} .$$

(b) Find the orthogonal trajectories to the family of circles $(x - a)^2 + y^2 = a^2$.

This is the family of circles with centers on the x-axis and tangent to the y-axis. The circles are given by

$$x^2 - 2ax + y^2 = 0 .$$

For the original family

$$y' = \frac{y^2 - x^2}{2xy}$$

and for the orthogonal family,

$$y' = \frac{2xy}{x^2 - y^2} = \frac{2\frac{y}{x}}{1 - (\frac{y}{x})^2} .$$

Set $v = \frac{y}{x}$, as in Part (a), to obtain

$$v' = \frac{v(1 + v^2)}{(1 - v^2)x} ,$$

hence,

$$\int \frac{1 - v^2}{v(1 + v^2)} dv = \int \frac{dx}{x} .$$

Set $v^2 = z$ to obtain

$$\begin{aligned}\int \frac{1 - v^2}{v(1 + v^2)}\,dv &= \frac{1}{2}\int \frac{1 - z}{z(1 + z)}\,dz \\ &= \frac{1}{2}\int \left(\frac{1}{z} - \frac{2}{1 + z}\right)dz \\ &= \frac{1}{2}\log \frac{|z|}{(1 + z)^2} + C \\ &= \frac{1}{2}\log \frac{v^2}{(1 + v^2)^2} + C \\ &= \log \frac{|v|}{1 + v^2} + C \\ &= \log \frac{\left|\frac{y}{x}\right|}{1 + \frac{y^2}{x^2}} + C \\ &= \log \frac{k|xy|}{x^2 + y^2}\end{aligned}$$

where $\log k = C$. Insert this result above to obtain

$$\frac{k|xy|}{x^2 + y^2} = |x| \ ,$$

hence,

$$x^2 + y^2 - 2by = 0 \ ,$$

or

$$x^2 + (y - b)^2 = b^2$$

where $b = \frac{k}{2}\operatorname{sgn} y$. In addition, $v = 0$ is a solution of the separable equation, and it is clear that $y = 0$ is also a solution. Thus the orthogonal trajectories consist of a similar family of circles with centers on the y-axis and tangent to the x-axis.

Solutions Miscellaneous Exercises 10

1. Integrate $\int a^x b^x dx$.

Use $a^x b^x = e^{x \log ab}$.

2. Integrate

$$\int \frac{e^{2x}}{\sqrt[4]{e^x + a^2}} dx .$$

Set $\sqrt[4]{e^x + a^2} = t \Longrightarrow e^x = t^4 - a^3, e^x dx = 4t^3 \, dt$.

$$I = \int \frac{(t^4 - a^2) + t^3}{t} dt = \frac{4}{7} t^7 - \frac{4a^2}{3} t^3$$

$$= \frac{4}{7} (e^x + a^2)^{7/4} - \frac{4a^2}{3} (e^x + a^2)^{3/4} .$$

3. Evaluate $I = \int_0^\infty \frac{x \, dx}{x^4 + x^2 + 1}$.

Set $x^2 = t$;

$$I = \frac{1}{2} \int_0^\infty \frac{dt}{t^2 + t + 1} = \frac{1}{\sqrt{3}} \arctan \frac{2t + 1}{\sqrt{3}} \Big|_0^\infty = \frac{\pi\sqrt{3}}{9} .$$

4. Evaluate $\int_0^a \frac{dx}{x + \sqrt{a^2 - x^2}}$.

Set $x = a \sin \theta$.

$$I = \int_0^{\pi/2} \frac{\cos \theta}{\sin \theta + \cos \theta} d\theta .$$

To avoid computation of the indefinite integral use the following special device.

Set $x = a\cos\theta$, too. Then

$$I = \int_0^{\pi/2} \frac{\sin\theta}{\sin\theta + \cos\theta}\,d\theta .$$

Thus

$$I = \frac{1}{2}\int_0^{\pi/2} \left(\frac{\sin\theta}{\sin\theta + \cos\theta} + \frac{\cos\theta}{\sin\theta + \cos\theta}\right)d\theta = \frac{\pi}{4} .$$

To obtain the indefinite integral, use

$$I = \int_0^{\pi/2} \frac{\cos\theta(\sin\theta + \cos\theta)}{(\sin\theta + \cos\theta)^2}\,d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{\cos 2\theta + 1 + \sin 2\theta}{1 + \sin 2\theta}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[\frac{\cos 2\theta}{1 + \sin 2\theta} + 1\right] = \frac{1}{2}\left[\log(1 + \sin 2\theta) + \theta\right]\Big|_0^{\pi/2} = \frac{\pi}{4} .$$

5. Prove that

$$\int_a^\infty \frac{dx}{x^4\sqrt{a^2 + x^2}} = \frac{2 - \sqrt{2}}{3a^4} .$$

Set $\alpha = at$. $\quad I = \frac{1}{a^4}\int_1^\infty t^4 \frac{dt}{\sqrt{1 + t^2}}$.

Set $t = \frac{1}{s}$. $\quad I = \frac{1}{a^4}\int_0^1 \frac{s^3dx}{\sqrt{1 + s^2}}$.

Set $s^2 = u$. $\quad I = \frac{1}{2a^4}\int_0^1 \frac{udu}{\sqrt{1 + u}} = \frac{1}{2a^4}\int_0^1\left(\sqrt{1 + u} - \frac{1}{\sqrt{1 + u}}\right)du$

$$= \frac{1}{2a^4}\left[\frac{2}{3}(1 + u)^{3/2} - 2(1 + u)^{1/2}\right|_0^1 .$$

6. Compare $\int_0^1 \frac{dx}{x^{1-\epsilon}}$ with $\frac{dx}{x^{1+\epsilon}}$.

Set $x = \frac{1}{v}$. Then

$$\int_0^1 \frac{dx}{x^{1-\epsilon}} = -\int_\infty^1 v^{1-\epsilon}\frac{dv}{v^2} = \int_1^\infty \frac{dv}{v^{1+\epsilon}} .$$

The two integrals are equal (the value is $\frac{1}{\epsilon}$) .

7. Evaluate $\int_{-1}^{1} \sqrt{\frac{1 + x}{1 - x}}\, dx$.

$$I = \int_{-1}^{1} \frac{1 + x}{\sqrt{1 - x^2}}\, dx = \arcsin x - \sqrt{1 - x^2}\ \Big|_{-1}^{1} = \pi .$$

8. Show that

$$2 \int_{0}^{\infty} G(t + \sqrt{t^2 + 1})dt = \int_{1}^{\infty} (1 + \frac{1}{t^2})G(t)dt ;$$

assuming that the integral exists.

Set $x = t + \sqrt{t^2 + 1}$. Then

$$dx = \left(1 + \frac{t}{\sqrt{t^2 + 1}}\right)dt$$

or

$$dt = \frac{\sqrt{t^2 + 1}}{t + \sqrt{t^2 + 1}}\, dx = \frac{\sqrt{t^2 + 1}}{x}\, dx .$$

Now

$$x + \frac{1}{x} = t + \sqrt{t^2 + 1} + \frac{1}{t + \sqrt{t^2 + 1}} = (t + \sqrt{t^2 + 1}) + (\sqrt{t^2 + 1} - t)$$

$$= 2\sqrt{t^2 + 1}$$

whence

$$dt = \frac{1}{2}(1 + \frac{1}{x^2})dx .$$

9. Prove that

$$\int_{0}^{\infty} (\sqrt{t^2 + 1} - t)^n dt = \frac{n}{n^2 - 1} \quad \text{provided } n > 1 .$$

$\sqrt{t^2 + 1} - t = \dfrac{1}{\sqrt{t^2 + 1} + t} \le \dfrac{1}{2t}$ and the integral is convergent. By Number 8,

$$\int_{0}^{\infty} \frac{1}{(t + \sqrt{t^2 + 1})^n}\, dt = \frac{1}{2} \int_{1}^{\infty} (1 + \frac{1}{t^2})t^{-n}dt$$

$$= \frac{1}{2}(\frac{1}{-n + 1}t^{-n+1} + \frac{1}{-n - 1}\, t^{-n-1})\Big|_{1}^{\infty} = \frac{n}{n^2 - 1} .$$

10. Compute

$$I = \int_0^\infty \frac{dx}{(x^2 + a^2)(x^2 + b^2)}, a > 0 , b > 0 , a \neq b .$$

$$\frac{1}{(x^2 + b^2)(x^2 + a^2)} = \frac{1}{a^2 - b^2}\left(\frac{1}{x^2 + b^2} - \frac{1}{x^2 + a^2}\right) .$$

$$I = \frac{1}{a^2 - b^2}\left(\frac{1}{b}\arctan\frac{x}{b} - \frac{1}{a}\arctan\frac{x}{a}\right)\Bigg|_0^\infty = \frac{\pi}{2ab(a + b)} .$$

11. Compute

$$\int_0^\infty \frac{x\,dx}{(x^2 + a^2)(x^2 + b^2)} , a > 0 , b > 0 .$$

Set $x^2 = t$.

$$I = \frac{1}{2}\int_0^\infty \frac{dt}{(t + a^2)(t + b^2)} ;$$

if $a \neq b$,

$$\frac{1}{(t + a^2)(t + b^2)} = \frac{1}{a^2 - b^2}\left(\frac{1}{t + b^2} - \frac{1}{t + a^2}\right) ,$$

$$I = \frac{1}{2(a^2 + b^2)} \log\frac{t + b^2}{t + a^2}\Bigg|_0^\infty = \frac{\log\frac{a}{b}}{a^2 - b^2} ;$$

if $a = b$,

$$I = \frac{1}{2}\int_0^\infty \frac{dt}{(t + a^2)^2} = \frac{1}{2a^2} .$$

12. Show that the integrals

(a) $\displaystyle\int_e^\infty \frac{dx}{x \log^\alpha x}$

(b) $\displaystyle\int_{e^e}^\infty \frac{dx}{x \log x\,(\log\log x)^\alpha}$

converge if $\alpha > 1$ and diverge if $\alpha \leq 1$.

In general, consider the iterated logarithmic and exponential functions defined by

$$L_1 = \log , L_{n+1} = \log L_n$$

$$E_1 = \exp , E_{n+1} = \exp E_n .$$

Note that $L_n'(x) = \frac{1}{xL_1(x)L_2(x) \dots L_{n-1}(x)}$. Consequently,

$$\int_{E_n(1)}^{\infty} \frac{dx}{[xL_1(x)L_2(x) \dots L_{n-1}(x)]\, L_n^{\alpha}(x)} = \lim_{x\sim\infty} \frac{L_n^{1-\alpha}(x)}{1-\alpha} + \frac{1}{\alpha - 1} ,$$

provided the limit exists. If $\alpha > 1$, then $\lim\limits_{x\sim\infty} L_n^{1-\alpha}(x) = 0$ and $I = \frac{1}{\alpha - 1}$. If $\alpha \leq 1$ then $L_n(x)$ increases without bound as x approaches ∞ and so also does $L_n^{1-\alpha}(x)$.

13. Prove that $\int_0^1 \sin \frac{1}{x}$ converges.

Substitute $x = \frac{1}{u}$ to obtain $I = \int_1^{\infty} \frac{\sin u}{u^2}\, du$ which is absolutely convergent by Theorem 10-6c. It follows from Exercises 10-6b, Number 2(d), by use of the inverse substitution $x = \frac{1}{u}$, that I converges.

14. (a) Prove that if f has a continuous derivative then

$$\int_1^x [\![t]\!]\, f'(t)dt = [\![x]\!]\, f(x) - \sum_{n=1}^{[\![x]\!]} f(n) \qquad ([\![t]\!] = \text{integral part of } t).$$

For $n \leq t < n + 1$, $[\![t]\!] = n$. Thus

$$\int_n^{n+1} [\![t]\!]\, f'(t)dt = n(f(n + 1) - f(n))$$

$$I = \sum_{k=1}^{n-1} \int_k^{k+1} [\![t]\!]\, f'(t)dt + \int_n^x [\![t]\!]\, f'(t)dt$$

$$= \sum_{n=1}^{[\![x]\!]-1} n[f(n + 1) - f(n)] + [\![x]\!]\, (f(x) - f([\![x]\!])$$

$$= \sum_{n=1}^{[\![x]\!]} (n - 1)f(n) - \sum_{n=1}^{[\![x]\!]} nf(n) + [\![x]\!]\, f(x) = [\![x]\!]f(x) - \sum_{n=1}^{[\![x]\!]} f(n) .$$

(b) Evaluate $\int_1^x \llbracket t \rrbracket t^2 dt$.

Using the formula of (a), with $f(t) = \frac{t^3}{3}$.

$$\int_1^x \llbracket t \rrbracket t^2 dt = \llbracket x \rrbracket \frac{x^3}{3} - \sum_{n=1}^{\llbracket x \rrbracket} \frac{n^3}{3}$$

$$= \llbracket x \rrbracket \frac{x^3}{3} - \frac{1}{12} \llbracket x \rrbracket^2 (\llbracket x \rrbracket + 1)^2 .$$

(c) Evaluate $\int_1^x \llbracket t^2 \rrbracket t \, dt$.

First make the substitution $t^2 = s$, then use (a) with $f(s) = s$.

$$\int_1^x \llbracket t^2 \rrbracket t \, dt \quad \frac{1}{2} \int_1^{x^2} \llbracket s \rrbracket ds = \frac{1}{2} (\llbracket x^2 \rrbracket x^2 - \sum_{n=1}^{\llbracket x^2 \rrbracket} n)$$

$$= \frac{1}{2} \llbracket x^2 \rrbracket x^2 - \frac{1}{4} \llbracket x^2 \rrbracket (\llbracket x^2 \rrbracket + 1) .$$

15. Find the maximum and the minimum of the function

$$F(x) = \int_0^x \frac{2t + 1}{t^2 - 2t + 2} dt \quad \text{in the interval } [-1,1] .$$

$F'(x) = 0$ if $x = -\frac{1}{2}$. Since $F'(x) < 0$ for $x < -\frac{1}{2}$ and $F'(x) > 0$ for $x > -\frac{1}{2}$ this will be the minimum point. Thus

$$F_{min} = -\int_{-1/2}^{0} \frac{2t + 1}{t^2 - 2t + 2} dt = F(-\tfrac{1}{2}) .$$

The maximum will occur at the end of the interval. Without computing, we can show that $F_{max} = F(1)$. Indeed,

$$F(1) - F(-1) = \int_0^1 \frac{2t + 1}{t^2 - 2t + 2} dt + \int_0^1 \frac{1 - 2t}{t^2 + 2t + 2} dt$$

(where the substitution $s = -t$ is employed to get the second integral.)

$$= \int_0^1 \frac{10t^2 + 4}{(t^2 - 2t + 2)(t^2 + 2t + 2)} dt > 0 .$$

Computation: $\displaystyle\int \frac{2t+1}{t^2-2t+2} = \log(t^2 - 2t + 2) + 3 \arctan (t - 1) + C$;

whence,

$$F_{\max} = F(1) = \frac{3\pi}{4} - \log 2 = 1.66$$

$$F_{\min} = F(-\tfrac{1}{2}) = -3(\arctan \tfrac{3}{2} - \tfrac{\pi}{4}) + \log \tfrac{13}{8} .$$

16. Compute

(a) $\displaystyle\int_{-\pi/4}^{\pi/4} \theta^{20} (\sin^{19} \theta - \sin^{17} \theta) d\theta$.

(b) $\displaystyle\int_{-2}^{2} \frac{x^{19} - x^{17} + x}{\cos^2 \frac{x}{2}}$.

(c) $\displaystyle\int_{-2}^{2} \sin^2 x \log \frac{3+x}{3-x} dx$.

All are zero. Odd functions.

17. Prove that $\displaystyle\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$.

Set $x = a + b - s$.

18. (a) Compute

$$\lim_{\epsilon \sim 0^+} \int_{a\epsilon}^{b\epsilon} \frac{dx}{x} , \quad b > a > 0 .$$

$$\lim_{\epsilon \sim 0^+} \int_{a\epsilon}^{b\epsilon} \frac{dx}{x} = \lim_{\epsilon \sim 0^+} \log \frac{b\epsilon}{a\epsilon} = \log \frac{b}{a} .$$

(b) Assume that f is continuous on $[0,\infty]$ and $\displaystyle\int_1^\infty \frac{f(x)}{x} dx$ exists. Show that

$$I = \int_0^\infty \frac{f(bx) - f(ax)}{x} dx = f(0) \log \frac{b}{a} .$$

$I = \lim_{\epsilon\sim 0^+} \int_{\epsilon}^{\infty} \frac{f(bx) - f(ax)}{x}\, dx$. Now, for $\epsilon < 0$,

$$\int_{\epsilon}^{\infty} \frac{f(bx)}{x}\, dx = \int_{\epsilon/b}^{\infty} \frac{f(s)}{s}\, ds \ , \int_{\epsilon}^{\infty} \frac{f(ax)}{x}\, dx = \int_{\epsilon/a}^{\infty} \frac{f(s)}{s}\, ds \ ,$$

both being convergent integrals which we can now subtract. Thus

$$\int_{\epsilon}^{\infty} \frac{f(bx) - f(ax)}{x}\, dx = \int_{\epsilon/b}^{\epsilon/a} \frac{f(x)}{x}\, dx \ .$$

Observe next that $f(0) \log \frac{b}{a} = \int_{\epsilon/b}^{\epsilon/a} \frac{f(0)}{x}\, dx$ and consider

$$J(\epsilon) = \int_{\epsilon/b}^{\epsilon/a} \frac{f(x) - f(0)}{x}\, dx \ .$$

It will suffice to show that $\lim_{\epsilon\sim 0^+} J(\epsilon) = 0$. Let α be any positive number. We will show that $|J(\epsilon)| \le \alpha \log \frac{b}{a}$ for sufficiently small ϵ . Since f is a continuous function, there exists δ such that $0 \le x \le \delta$ implies $|f(x) - f(0)| \le \alpha$. Let $\epsilon \le a\delta$. Then $\frac{\epsilon}{b} < \frac{\epsilon}{a} < \delta$ and thus

$$|J(\epsilon)| = \left| \int_{\epsilon/b}^{\epsilon/a} \frac{f(x) - f(0)}{x}\, dx \right| \le \int_{\epsilon/b}^{\epsilon/a} \frac{|f(x) - f(0)|}{x}\, dx$$

$$\le \alpha \int_{\epsilon/b}^{\epsilon/a} \frac{dx}{x} = \alpha \log \frac{b}{a} \ .$$

19. By evaluating $I = \int_0^1 (1 - t^2)^n dx$, n a positive integer in two different ways show that

$$\frac{\binom{n}{0}}{1} - \frac{\binom{n}{1}}{3} + \frac{\binom{n}{2}}{5} \dots \frac{(-1)^n\binom{n}{n}}{2n + 1} = \frac{2^{2n}}{\binom{2n}{n}(2n + 1)} \ .$$

Using the binomial theorem, we get $(1 - t^2)^n = \binom{n}{0} - \binom{n}{1}t^2 + \binom{n}{2}t^3 - \dots$, and on integrating the polynomial we get the left side. Next, set $t = \sin x$. Then $I = \int_0^{\pi/2} \cos^{2n+1} x\, dx = \frac{(2^n n!)}{(2n + 1)!} = \frac{2^{2n}}{\binom{2n}{n}(2n + 1)}$ (see Example 10-7d).

20. Let $\phi_0(x) = \phi(x)$ be a continuous function defined on $[0,x]$. Define $\phi_n(x)$ by

$$\phi_n(x) = \int_0^x \phi_{n-1}(y)dy \ , \ n = 1,2,\ldots \ .$$

Show that

$$\phi_{n+1}(x) = \frac{1}{n!}\int_0^x \phi(y)(x - y)^n dy \ .$$

Proof by induction. Set

$$I(x) = \frac{1}{n!}\int_0^x \phi(y)(x - y)^n dy \ .$$

Show that

$$I(x) = \frac{1}{(n - k)!}\int_0^x \phi_k(y)(x - y)^{n-k}dy$$

for $0 \le k \le n - 1$. By definition, the statement is true for $k = 0$. If true for any value of $k < n$, then integrate by parts with $u(y) = (x - y)^{n-k}$, $dv = \phi_k(y)dy$ and $v = \phi_{k+1}(y)$. Thus

$$I(x) = \frac{1}{(n-k)!}[\phi_k(y)(x-y)^{n-k}\Big|_{y=0}^{x} + (n-k)\int_0^x \phi_{k+1}(y)(x-y)^{n-k-1}dy]$$

where the first term vanishes since $\phi_k(0) = 0$ for $k \ge 1$, and $(x - y)^{n-k} = 0$ for $k < n$. Set $k = n - 1$ above to obtain

$$I(x) = \int_0^x \phi_n(y)(x - y)^n dy = \phi_{n+1}(x)$$

which was to be proved.

Alternate Solution: Use the binomial theorem to obtain

$$I(x) = \frac{1}{k!}\int_0^x \phi(y)\left[\sum_{\nu=0}^{k} (-1)^{\nu}\binom{k}{\nu}x^{k-\nu}y^{\nu}\right]dy$$

$$= \frac{1}{k!}\sum_{\nu=0}^{k} (-1)^{\nu}\binom{k}{\nu}x^{k-\nu}\int_0^x \phi(y)y^{\nu}dy \ ,$$

whence

$$I'(x) = \frac{1}{(k-1)!}\sum_{\nu=0}^{k-1} (-1)^{\nu}\binom{k}{\nu}\frac{k-\nu}{k}\, x^{k-\nu-1}\int_0^x \phi(y)y^{\nu}dy$$

$$+ \frac{1}{k!}\sum_{\nu=0}^{k} (-1)^{\nu}\binom{k}{\nu} x^{k-\nu}\,\phi(x)x^{\nu}$$

$$= \frac{1}{(k-1)!}\int_0^x \phi(y)\sum_{\nu=0}^{k-1} (-1)^{\nu}\binom{k}{\nu}\frac{k-\nu}{k}\, x^{k-1-\nu}\, y^{\nu}\, dy$$

$$+ \frac{1}{k!}\sum_{\nu=0}^{k} (-1)^{\nu}\binom{k}{\nu}\phi(x)$$

$$= \frac{1}{(k-1)!}\int_0^x \phi(y)(x-y)^{k-1}dy + \frac{1}{k!}(1-1)^k x^k\phi(x)$$

(since $\binom{k}{\nu}\frac{k-\nu}{k} = \binom{k-1}{\nu}$.)

$$= \phi_k(x) + 0 ,$$

by induction.

21. Prove that for all positive integers N ,

$$0 \le \sum_{n=1}^{N} \frac{a}{a^2+n^2} < \frac{\pi}{2} .$$

If $n-1 \le x \le n$ then $\frac{a}{a^2+n^2} \le \frac{a}{a^2+x^2}$, and

$$\frac{a}{a^2+n^2} = \frac{a}{a^2+n^2}\int_{n-1}^{n} dx \le \int_{n-1}^{n}\frac{adx}{a^2+x^2} .$$

Thus

$$\sum_{n=1}^{N}\frac{a}{a^2+n^2} \le \sum_{n=1}^{N}\int_{n-1}^{n}\frac{adx}{a^2+x^2} \le \int_0^N \frac{adx}{a^2+x^2} < \int_0^{\infty}\frac{adx}{a^2+x^2} = \frac{\pi}{2} .$$

22. (a) Let L and M be linear operators and define the sum $L + M$ as the operator

$$L + M : u \longrightarrow L[u] + M[u] .$$

Verify that $L + M$ is linear.

$$\begin{aligned}(L + M)[au + bv] &= L[au + bv] + M[au + bv] \\ &= (aL[u] + bL[v]) + (aM[u] + bM[v]) \\ &= a(L[u] + M[u]) + b(L[v] + M[v]) \\ &= a(L + M)[u] + b(L + M)[v] .\end{aligned}$$

(b) Show that linear operators satsify the distributive laws:

$$L(M + N) = LM + LN$$
$$(L + M)N = LN + MN$$

$$\begin{aligned}L(M + N)[u] &= L(M[u] + N[u]) && \text{(definition of } M + N\text{)} \\ &= LM[u] + LN[u] && \text{(by linearity)} \\ &= (LM + LN)[u] && \text{(definition).}\end{aligned}$$

Same method applied to other distributive law.

(c) Let f be any real-valued function. The multiplication operator $f\cdot$ is defined by

$$f \cdot [u] = f \cdot u .$$

Verify that $f\cdot$ is linear.

This is ordinary multiplication of a function by f.

$$\begin{aligned}f \cdot (au + bv) &= f \cdot (au) + f \cdot (bv) && \text{(distributive law)} \\ &= af \cdot u + bf \cdot v && \text{(associative law).}\end{aligned}$$

(d) For real-valued functions a_n, a_{n-1}, $\cdots$, a_0 show that the differential operator L of n-th order given by

$$L = a_n \cdot D^n + a_{n-1} \cdot D^{n-1} + \ldots + a_0$$

is linear.

L is formed by composition and addition of linear operators.

23. (a) Prove for the second order linear differential operator L defined by $L = D^2 + p \cdot D + q$ that if u is a nontrivial solution of the homogeneous equation $L[u] = 0$, there is another solution of the form $u \cdot v$ where v' satisfies a linear differential equation $M[v'] = 0$ and M is at most of first order.

If $L[u] = 0$, then for any function v,

$$L[u \cdot v] = v \cdot L[u] + u \cdot v'' + (2u' + p) \cdot v' = M[v'] ,$$

where $M = u \cdot D + 2u' + p$. Thus $L[u \cdot v] = 0$ if and only if $M[v'] = 0$. In a neighborhood where $u(x) \neq 0$ there exists a nontrivial solution v' of the first order equation. Since v' is not identically zero, its integral v is not a constant function hence $u \cdot v$ is a new solution, linearly independent of u (see No. 24).

(b) Prove, in general, for the n-th order linear differential operator L defined by $L = \sum_{k=0}^{n} a_k(x) D_x^k$, $a_n(x) \neq 0$ that if u is a nontrivial solution of the homogeneous equation $L[u] = 0$, there is another solution of the form $u \cdot v$ where v' satisfies a linear differential equation $M[v'] = 0$ where M is at most of $(n - 1)$-th order.

$$\begin{aligned}
L[u \cdot v] &= \sum_{k=0}^{n} a_k \cdot D^k[u \cdot v] \\
&= \sum_{k=0}^{n} a_k \cdot \sum_{r=0}^{k} \binom{k}{r} D^r[v] \cdot D^{k-r}[u] \\
&= \sum_{k=0}^{n} a_k \cdot \left\{ v \cdot D^k[u] + \sum_{r=1}^{k} \binom{k}{r} D^r[v] \cdot D^{k-r}[u] \right\} \\
&= v \cdot \sum_{k=0}^{n} a_k \cdot D^k[u] + \sum_{k=1}^{n} \sum_{r=1}^{k} a_k \cdot \binom{k}{r} D^r[v] \cdot D^{k-r}[u] \\
&= v \cdot L[u] + \sum_{r=1}^{n} \left[\sum_{k=1}^{n} \binom{k}{r} a_k \cdot D^{k-r}[u] \right] \cdot D^r[v] \\
&= 0 + M[v'] .
\end{aligned}$$

Where $M = \sum_{r=0}^{n-1} b_{r+1} \cdot D^r$ is an operator of order at most $a - 1$,

and $b_{r+1} = \sum_{k=r}^{n} \binom{k}{r} a_k D^{k-r}[u]$.

(c) Find an n-parameter family of solutions of the differential equation

$$L[u] = (D - a)^n u = 0 .$$

Since $(D - a)u = 0$ has the solution $u = e^{ax}$ we know that this is also a solution of the original equation, $L[u] = 0$. Attempt to find another solution of the form $u \cdot v$. From

$$(D_x - a)[e^{ax}v(x)] = e^{ax}v'(x)$$

it follows at once that

$$(D_x - a)^n[e^{ax}v(x)] = e^{ax}v^{(n)}(x) .$$

Hence $L[e^{ax}v(x)] = 0$ if and only if $v^{(n)}(x) = 0$. Thus $v^{(n)}$ may be any polynomial of degree up to $n - 1$. Thus an n-parameter family of solutions is

$$e^{ax}[c_0 + c_1x + \ldots + c_{n-1}x^{n-1}] .$$

(d) Obtain the general solution of

$$(x^2 + 1)y'' + xy' - 4y = 0 .$$

(Hint: Try to find a particular polynomial solution.)

Attempt a solution in the form

$$y = \sum_{\nu=0}^{n} a_\nu x^\nu .$$

Then

$$(x^2 + 1) \sum_{\nu=0}^{n} \nu(\nu - 1)a_\nu x^{\nu-2} + x \sum_{\nu=0}^{n} \nu a_\nu x^{\nu-1} - 4 \sum_{\nu=0}^{n} a_\nu x^\nu .$$

Collect like powers and equate the coefficients to zero; then for the coefficient of x^ν

$$(\nu + 1)(\nu + 2)a_{\nu+2} + (\nu(\nu - 1) + \nu - 4)a_\nu = 0$$

hence,

$$a_{\nu+2} = \frac{(4 - \nu^2)a_\nu}{(\nu + 1)(\nu + 2)} .$$

Take $a_0 = 1$, $a_1 = 0$. Then $a_{2k+1} = 0$ for all k , $a_2 = 2$, and $a_{2n} = 0$ for $n > 1$. Thus there is a solution

$$y = 2x^2 + 1 .$$

Now seek a solution in the form

$$u(x) = (2x^2 + 1)v(x) .$$

Take

$$\begin{aligned} L[u(x)] &= (x^2 + 1)[(2x^2 + 1)v''(x) + 8xv'(x)] + x(2x^2 + 1)v'(x) \\ &= (2x^4 + 3x^2 + 1)v''(x) + (2x^3 + 9x)v'(x) \\ &= 0 . \end{aligned}$$

This is a linear first order equation for $v'(x)$ and has the solution

$$v'(x) = e^{I(x)}$$

where

$$\begin{aligned} I(x) &= -\int_0^x \frac{2t^3 + 9t}{(t^2 + 1)(2t^2 + 1)} dt \\ &= -\int_0^x \frac{16t}{2t^2 + 1} - \frac{7t}{t^2 + 1} dt \\ &= -4 \log (2t^2 + 1) + \frac{7}{2} \log (t^2 + 1) . \end{aligned}$$

Thus

$$v'(x) = \frac{(t^2 + 1)^{7/2}}{(2t^2 + 1)^4} .$$

This may be explicitly integrated again to give $v(x)$ by either substitution $t = \tan \theta$ or $t = \sinh z$. Then

$$u(x) = (2x^2 + 1) \int_0^x \frac{(t^2 + 1)}{(2t^2 + 1)^4} dt .$$

24. Two functions u and v are called linearly independent on an interval I if

$$\alpha u(x) + \beta v(x) = 0 \text{ for all } x \text{ in } I$$

implies that α and β are both zero; i.e., if u and v are not proportional. Show that if two linearly independent solutions of a homogeneous second order linear equation exist in the neighborhood of x_0, then the initial value problem (13) of Section 10-8 can be solved.

Set

$$w^*(x) = u(x)v'(x) - v(x)u'(x) .$$

By the same derivation as for Formula (18c), Section (18c),

$$w^*(x) = w^*(x_0) \exp\left\{-\int_{x_0}^{x} p(t)dt\right\} .$$

Thus $w^*(x)$ is either always non-zero or always zero, with $w^*(x_0)$. Now, $u(x)$ is not identically zero, for then

$$1 \cdot u + 0 \cdot v = 0$$

and the two functions are linearly dependent. Thus there exists a point x_1, where $u(x_1) \neq 0$ which has a neighborhood where $u(x) \neq 0$. In this neighborhood consider

$$\frac{w^*(x)}{u(x)} = D_x\left(\frac{v(x)}{u(x)}\right) .$$

If $w^*(x_0) = 0$, then $w^*(x) = 0$ identically, hence $\frac{v(x)}{u(x)}$ is constant, and the two functions are linearly dependent on the neighborhood of x_1; $au(x) + v(x) = 0$. But $z = au + v$ is a solution of $L[u] = 0$ and satisfies the initial condition $z(x_1) = z'(x_1) = 0$. Hence z is the trivial solution and $au + v = 0$ identically; i.e., linear independence is violated. Consequently

$$w^*(x_0) = u(x_0)v'(x_0) - v(x_0)u'(x_0) \neq 0 .$$

From this conclusion, it follows that there exist constants c_1 and c_2 such that

$$\begin{cases} c_1u(x_0) + c_2v(x_0) = y_0 , \\ c_1u'(x_0) + c_2v'(x_0) = y_0' , \end{cases}$$

namely,

$$c_1 = \frac{y_0 v'(x_0) - y_0'(x_0)}{w^*(x_0)} \quad , \quad c_2 = \frac{y_0' u(x_0) - y_0 u'(x_0)}{w^*(x_0)} \quad .$$

Thus, it is possible to find a solution

$$z = c_1 u + c_2 v$$

which satisfies arbitrarily prescribed initial conditions

$$z(x_0) = y_0 \quad , \quad z'(x_0) = y_0' \ .$$

25. Corresponding to the three classes of second order homogeneous linear equation with constant coefficients, Section 10-8, we found the following pairs of solutions $\{e^{\alpha x}, e^{\beta x}\}$ where $\alpha \neq \beta$ $\{e^{ax}, xe^{ax}\}$, and $\{e^{ax} \cos \omega x, e^{ax} \sin \omega x\}$ where $\omega \neq 0$.
Prove that each of these pairs of solutions is linearly independent.

If two functions u_1, u_2 are linearly dependent, then there exist constants c_1 and c_2, not both zero, say $c_1 \neq 0$, for which

$$c_1 u_1 + c_2 u_2 = 0 \ .$$

Differentiate to obtain also

$$c_1 u_1' + c_2 u_2' = 0 \ .$$

Eliminate c_2 : multiply in the first equation by u_2', the second by u_2, and subtract:

$$c_1 u_1 \cdot u_2' + c_2 u_2 \cdot u_2' - c_1 u_1' \cdot u_2 - c_2 u_2' \cdot u_2$$

or

$$c_1(u_1 \cdot u_2' - u_1' \cdot u_2) = 0 \ .$$

Since $c_1 \neq 0$; then

$$w = u_1 \cdot u_2' - u_1' \cdot u_2 = 0 \ ;$$

i.e., if the functions are linearly dependent, then $w = 0$. Consequently if for some x, $w(x) \neq 0$ the functions are linearly independent. Apply this criterion to the given pairs of functions: if $u_1(x) = e^{\alpha x}$, $u_2(x) = e^{\beta x}$, then $w(x) = (\beta - \alpha)e^{(\alpha+\beta)x} \neq 0$: if $u_1 = e^{ax}$, $u_2 = xe^{ax}$, then $w(x) = e^{2ax} \neq 0$: if $u_1 = e^{ax} \sin \omega x$, $u_2 = e^{ax} \cos \omega x$, then $w(x) = \omega e^{2ax} \neq 0$.

26. Let u be any nontrivial solution of the homogeneous second order linear equation

$$L[u] = D^2u + p \cdot Du + q \cdot u = 0 .$$

Obtain a second solution v in the form $v = u \cdot z$. Show that u and v are linearly independent.

By the method of Number 23

$$L[u \cdot z] = z \cdot L[u] + u \cdot D^2z + (2u' + p \cdot u) \cdot z' ,$$

hence for a solution in the form $v = u \cdot z$

$$D^2z'' + (2 \frac{u'}{u} + p) \cdot z' = 0$$

where we may divide by u in any interval on which $u(x) \neq 0$. This is a first order linear equation for z' which has the solution

$$z'(x) = \exp\left\{-\int_{x_0}^{x} \left(p(t) + \frac{2u'(t)}{u(t)}\right)dt + C\right\}$$

$$= C \exp\left\{-2 \log u(x) - \int_{x_0}^{x} p(t)dt\right\}$$

$$= C \frac{w(x)}{u(x)^2}$$

where take $C = 2 \log u(x_0)$, $w(x) = \exp - \int_{x_0}^{x} p(t)dt$. Integrate again to obtain

$$v(x) = u(x)z(x) = u(x)\int_{x_0}^{x} \frac{w(t)}{u(t)^2} dt .$$

Now use Number 21 to show linear independence

$$u(x)v'(x) - v(x)u'(x) = u(x) \left[u'(x)z(x) - u(x) \frac{w(x)}{u(x)^2}\right] - u'(x)u(x)z(x)$$

$$= w(x) \neq 0 .$$

27. Let u and v be any linearly independent solutions of the second order linear homogeneous equation. Verify that the Green's functions is not affected if u and v replace the fundamental set $\{\phi,\psi\}$; namely

$$G(x,\xi) = \frac{\psi(x)\phi(\xi) - \phi(x)\psi(\xi)}{\phi(\xi)\psi'(\xi) - \psi(\xi)\phi'(\xi)}$$

$$= \frac{v(x)u(\xi) - u(x)v(\xi)}{u(\xi)v'(\xi) - v(\xi)u'(\xi)} .$$

The Green's function is uniquely determined by the conditions (i) $D_x^2G + p\cdot DG + q\cdot G = 0$, (ii) $G(\xi,\xi) = 0$, (iii) $D_xG(x,\xi)\Big|_{x=\xi} = 1$. Since the given function is a linear combination of solutions u and v of the reduced equation, (i) is satisfied. Furthermore (ii) and (iii) are clearly satisfied.

Alternatively, let the initial values at x_0 of u and v be given by

$$\begin{cases} u(x_0) = a \\ u'(x_0) = b , \end{cases} \qquad \begin{cases} v(x_0) = c \\ v'(x_0) = d . \end{cases}$$

Since u and v are linearly independent, it follows by Number 24 that

$$w^*(x_0) = u(x_0)v'(x_0) - u'(x_0)v(x_0) = ad - bc \neq 0 .$$

From the uniqueness theorem,

$$u(x) = a\phi(x) + b\psi(x) \quad , \quad v(x) = c\phi(x) + d\psi(x) ,$$

hence,

$$\frac{v(x)u(\xi) - u(x)v(\xi)}{u(\xi)v'(\xi) - v(\xi)u'(\xi)} = \frac{(ad - bc)[\psi(x)\phi(\xi) - \phi(x)\psi(\xi)]}{(ad - bc)[\phi(\xi)\psi'(\xi) - \psi(\xi)\phi'(\xi)]}$$

$$= G(x,\xi) .$$

28. Discuss the solution of equations of the type $y'' + f(y) = 0$.

In the equation

$$y'' + f(y) = 0$$

multiply by y' to obtain

$$y''y' + f(y)y' = 0$$

and integrate to obtain

$$\frac{1}{2}(y')^2 - \int_{y_0}^{y} f(s)ds = C \ .$$

Set $F(y) = \int_{y_0}^{y} f(s)ds$. Then if C is chosen so that $C - F(y) > 0$, $y' = \pm \sqrt{2C - 2F(y)}$, or

$$\frac{y'}{\sqrt{2C - 2F(y)}} = \pm 1 \ .$$

Set $G(y) = \int_{y_0}^{y} \frac{dy}{\sqrt{2C - 2F(y)}}$ to obtain the solutions

$$y = H(x + A) \quad , \quad y = H(-x + A)$$

where H indicates the inverse of G if it exists.

Teacher's Commentary

Appendix 6

TC. A6-1, Integration by Summation Techniques.

Once the student learns the Fundamental Theorem he may come to believe that the original conception of integral as the limit of a sum is not useful for analysis or computation. In this section it is shown that the formal integrals of polynomials and of the circular functions sin and cos can be obtained directly from the definition by summation techniques. This is something of a tour-de-force, but many students find the approach illuminating. Of course, summation remains valuable as a method of getting numerical estimates.

Solutions Exercises A6-1

1. In subsection (i) the text states that it follows "at once" from the inequality (6) that

$$\lim_{h \sim 0} Q(h) = 0 .$$

Actually, what theorems on limits are being used?

For $\lim_{h \sim 0} 0 = 0$, Theorem 3-4a.

For $\lim_{h \sim 0} Ca^r h = 0$, Theorem 3-4b or Example 3-3a.

From $0 \le |Q(h)| \le Ca^r h$, the Squeeze Theorem (Corollary 2 to Theorem 3-4f) to prove that

$$\lim_{h \sim 0} |Q(h)| = 0 .$$

2. Show simply, without repeating the argument of the text, that the lower sum over σ , $L = \sum_{k=1}^{n} x_{k-1}{}^r (x_{k-1} - x_k)$, also has the limit (7).

Since $L = U - h^{r+1} n^r = U - h(b - a)^r$, $\lim_{h \sim 0} L = \lim_{h \sim 0} U = \frac{a^{r+1}}{r + 1}$.

3. Employ Equation (8) of Section A3-2(ii) to obtain $\int_0^a \sin x \, dx$ for $0 < a \leq \frac{\pi}{2}$.

Replace a by $h = \frac{a}{n}$ in Equation (8) of Section A3-2(ii), and note that

$$\sin \frac{(n+1)h}{2} \sin \frac{nh}{2} = \cos \frac{h}{2} - \cos (n + \tfrac{1}{2})h$$

to obtain

$$\sum_{k=1}^{n} \sin kh = \frac{\cos \frac{h}{2} - \cos(n + \frac{1}{2})h}{2 \sin \frac{h}{2}} .$$

Since $\sin x$ is increasing on $[0,a]$ the upper and lower Riemann sums for a subdivision of the interval into n equal parts are given by

$$U = \sum_{k=1}^{n} h \sin kh \quad \text{and} \quad L = \sum_{k=1}^{n} h \sin (k-1)h .$$

Note that $L = U - h \sin a$. Insert $n = ah$ in the formula for $\sum_{k=1}^{n} \sin kh$ to obtain for the upper sum,

$$U = \frac{\frac{1}{2} h}{\sin \frac{1}{2} h} [\cos \tfrac{1}{2} h - \cos (a + \tfrac{1}{2} h)] .$$

Use the continuity of the cosine and

$$\lim_{h \sim 0} \frac{\frac{1}{2} h}{\sin \frac{1}{2} h} = 1$$

to obtain

$$\lim_{h \sim 0} U = 1 - \cos a = \lim_{h \sim 0} L ;$$

whence

$$\int_0^a \sin x \, dx = 1 - \cos a .$$

TC. A6-2. Existence of the Integral.

This section bases the existence theory of the integral on the Separation Axiom and is left in the appendix for that reason. The material through Lemma A6-2c is made very simple by use of the Separation Axiom and it is recommended for inclusion if completeness of the real number system (A1-5) has been discussed. The remaining material, beginning with Lemma A6-2d, is denoted to demonstrating the equivalence of the present concept of integral to that of Riemann integral and is somewhat more demanding.

Other books which define integral in terms of upper and lower sums generally use the sharpest possible estimates, namely, the Darboux upper and lower sums for which

$$M_k = \mathrm{lub}\{f(x) : x \in [x_{k-1},x_k]\}$$

and $$m_k = \mathrm{glb}\{f(x) : x \in [x_{k-1},x_k]\}$$, respectively. The Darboux sums are useful in the proof that an integrable function is Riemann integrable (Exercises A6-2, No. 3), but by the use of all upper and lower sums we eliminate some of the details of proof which the Darboux sums require.

Solutions Exercises A6-2

1. Let f be a function which takes on a maximum and minimum on every closed interval (e.g., f could be a continuous function, or monotone). Let $U^*(\sigma)$ and $L^*(\sigma)$ be the upper and lower Riemann sums obtained by using the maximum and minimum values of $f(x)$ as the appropriate bounds in each interval of the subdivision.

   Let σ_1 and σ_2 be any partitions of $[a,b]$. Prove for the joint subdivision $\sigma = \sigma_1 \cup \sigma_2$ that

   $$U^*(\sigma_1) \geq U^*(\sigma) \geq L^*(\sigma) \geq L^*(\sigma_2) .$$

   In other terms, by adding new points to a subdivision we may reduce the difference between the upper and lower Riemann sums, and we cannot increase it.

   Let $\sigma_1 = \{x_0,x_1,\ldots,x_n\}$ and consider the partition τ of the subinterval, $\tau = \{u_0,u_1,\ldots,u_p\}$ where $u_0 = x_{k-1}$ and $u_p = x_k$, and $u_1 , u_2 , \ldots , u_{p-1}$ are those points (if any) of σ_2 which lie in the interior of $[x_{k-1},x_k]$. If M_k is the maximum of $f(x)$ in $[x_{k-1},x_k]$, then for the maximum r_j of $f(x)$ in any subinterval $[u_{j-1},u_j]$ we have $r_j \leq M_k$. It follows that

$$M_k(x_k - x_{k-1}) \geq \sum_{j=1}^{p} r_j(u_j - u_{j-1}) .$$

Thus we have compared the k-th term in $U^*(\sigma_1)$ with the sum of those terms in $U^*(\sigma)$ which correspond to the subintervals of $[x_{k-1},x_k]$. It follows on addition that

$$U^*(\sigma_1) \geq U^*(\sigma) .$$

In the same way, show that

$$L^*(\sigma_2) \leq L^*(\sigma) .$$

Since $U^*(\sigma)$ and $L^*(\sigma)$ are upper and lower sums for the same partition we have

$$L^*(\sigma) \leq U^*(\sigma)$$

from which the result now follows.

2. Show that if f is Riemann integrable (Section 6-4) over $[a,b]$, then f is bounded on $[a,b]$.

Since $\lim\limits_{\nu(\sigma)\to 0} R(\sigma)$ exists, then for any positive ϵ there exists a subdivision σ of $[a,b]$ into n equal parts such that

(i) $$|h \sum_{k=1}^{n} f(\xi_k) - I| < \epsilon$$

no matter how ξ_k is chosen in $[x_{k-1},x_k]$. Let ξ_1 , ... , ξ_n be any preliminary choice of the ξ_k and let N be any positive integer.

If $f(x)$ is unbounded on $[a,b]$ we can always find a point ξ in $[a,b]$ such that

$$h|f(\xi)| > N(1 + |I| + h\cdot\sum_{k=1}^{n} |f(\xi_k)|) .$$

Since ξ lies in $[a,b]$ it must lie in at least one of the subintervals, say $[x_{i-1},x_i]$. We now replace ξ_i by ξ , and with this replacement (i) must still be satisfied. However

$$\left|h\sum_{k=1}^{n} f(\xi_k) - I\right| = \left|hf(\xi) + h\sum_{k\neq 1} f(\xi_k) - I\right|$$

$$\geq h|f(\xi)| - \left|h\sum_{k\neq 1} f(\xi_k) - I\right|$$

$$> \left(N + |I| + h\sum_{k=1}^{n} |f(\xi_k)|\right) - \left[h\sum_{k\neq 1} |f(\xi_k)| + |I|\right]$$

$$\geq N + h|f(\xi_1)|$$

$$\geq N .$$

Since N is any positive integer we have an immediate contradiction if we take $\epsilon = \frac{1}{2}$ in (1).

3. Let f be integrable over $[a,b]$ and let R denote a Riemann sum corresponding to a partition σ of $[a,b]$. We have proved (Theorem 6-3c) that if f has an integral I then

$$I = \lim_{\nu(\sigma)\to 0} R .$$

Prove conversely that if the limit of the Riemann sums exists then it is the integral of f over $[a,b]$. (Hint: Show first for any partition and positive ϵ that there exist at least one Riemann sum R and one upper sum U over σ such that $U - R < \epsilon$.)

The student who independently obtains a solution to this problem should be commended for his achievement.

Since the function f is Riemann integrable, given $\hat{\epsilon} > 0$. There exists a partition σ such that for all Riemann sums R over σ

$$(1) \qquad |R - J| < \hat{\epsilon}$$

where J is the limit of Riemann sums over τ as $\nu(\tau)$ approaches 0. Let $\sigma = \{x_0, x_1, \ldots, x_n\}$ where

$$x_0 < x_1 < x_2 < \ldots < x_n .$$

Let M_k be the least upper bound of $f(x)$ on $[x_{k-1}, x_k]$. By Corollary 1 to the Least Upper Bound Principle, there exists a value $f(\xi_k)$ for some ξ_k in $[x_{k-1}, x_k]$ such that $M_k - f(\xi_k) < \bar{\epsilon}$.

Introduce an upper sum U and Riemann sum R_1 over σ :

$$U = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) \text{ , } R_1 = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \text{ ;}$$

then

$$U - R_1 = \sum_{k=1}^{n} (M_k - f(\xi_k))(x_k - x_{k-1}) < \sum_{k=1}^{n} \bar{\epsilon}(x_k - x_{k-1}) \text{ ;}$$

whence,

$$U - R_1 < \bar{\epsilon}\ (b - a) \text{ .}$$

A parallel argument yields a lower sum L and Riemann sum R_2 such that

$$R_2 - L < \bar{\epsilon}(b - a) \text{ .}$$

From (i),

$$|R_1 - J| < \hat{\epsilon} \text{ and } |R_2 - J| < \hat{\epsilon} \text{ ;}$$

whence,

$$|R_2 - R_1| < 2\hat{\epsilon} \text{ .}$$

We have

$$\begin{aligned}(U - R_1) + (R_2 - L) &= (U - L) + (R_2 - R_1) \\ &< 2\bar{\epsilon}\ (b - a) \text{ ;}\end{aligned}$$

whence,

$$\begin{aligned}U - L &< 2\bar{\epsilon}\ (b - a) + |R_2 - R_1| \\ &< 2\bar{\epsilon}\ (b - a) + 2\hat{\epsilon} \text{ .}\end{aligned}$$

Given any positive ϵ , we require $\hat{\epsilon} = \frac{\epsilon}{4}$ and $\bar{\epsilon} = \frac{\epsilon}{4(b - a)}$ so that

$$U - L < \epsilon \text{ .}$$

Since we can always find an upper and a lower sum closer together than ϵ , it follows that f is integrable over $[a,b]$.

4. Consider the function f defined on $[0,1]$ by

$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ 1, & x \text{ rational} . \end{cases}$$

Prove that the integral of f does not exist.

Every interval contains both rational and irrational points. Consequently the maximum of $f(x)$ in every interval is 1 and the minimum is 0. For any partition σ of $[0,1]$ we have for any upper sum U and any lower sum L,

$$U - L = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1})$$

where $M_k \geq 1$ and $m_k \leq 0$. It follows that

$$M_k - m_k \geq 1$$

and

$$U - L \geq \sum_{k=1}^{n} (x_k - x_{k-1})$$

$$\geq 1$$

for all upper and lower sums.

The criterion of Theorem 6-3a cannot be satisfied and f is not integrable over $[0,1]$.

5. Consider the function f defined on $[0,1]$ by

$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ \frac{1}{t}, & x \text{ rational}, \ x = \frac{s}{t} \text{ in lowest terms.} \end{cases}$$

Prove that the integral of f over $[0,1]$ exists and find its value.

As an upper bound on the number of rational points with a given denominator $t > 1$ and numerator s relatively prime to t and less than t we have $t - 1$. For a fixed positive integer q introduce the partition

$$\sigma = \{\frac{1}{2q^2}, 1 - \frac{1}{2q^2}, \frac{s}{t} \pm \frac{1}{(t-1)2q^2}, (t = 2, \ldots, q)\}.$$

Thus each point of the form $\frac{s}{t}$, $t > 1$, is contained in a subinterval of the partition of length at most $\frac{1}{(t-1)q^2}$.

Since $f(\frac{s}{t}) = \frac{1}{t} < 1$, we see that the maximum contribution to the upper sum of the rational points with denominator t , $(t \geq 1)$ is $\frac{1}{q^2}$. Taking the sum of the contributions of all rational points with denominators $t = 1, 2, \ldots, q$, we obtain a contribution no greater than $\frac{q}{q^2} = \frac{1}{q}$. In the remaining intervals of the subdivision we have $f(x) < \frac{1}{q}$ and since the total length of the remaining subintervals is at most 1 , we have a contribution to the upper sums of no more than $\frac{1}{q}$. In this way we have found an upper sum U over σ for which $U < \frac{2}{q}$. Since we can always take the lower sum $L = 0$ and q may be any positive integer whatever, it follows that the integral exists and has the value 0 .

6. Give an example of a nonintegrable function fg where f and g are each integrable.

Let g be the function defined in Number 5 and take $f = \text{sgn}$. The function fg is then the given function of Number 4.

Teacher's Commentary

Appendix 7

INTEGRABILITY OF CONTINUOUS FUNCTIONS

This appendix together with Section A6-2 beginning with Lemma A6-2d is appropriate "research" material for the exceptionally able student who would like to pursue the theory of integration beyond the confines of the text. Most students will need some assistance in working through this material.

This appendix contains the existence theory for the integrals of continuous functions. It is based on the uniform continuity of a continuous function on a closed interval. The concept, uniform continuity, is not named here because it is needed only this once and, if it were introduced by name, the idea would have to be expounded carefully to bring out the distinction from continuity.

The Heine-Borel Principle is used for the proof because it is particularly useful for the derivation of global properties from local ones such as uniform continuity from continuity or boundedness from continuity (Exercises A7-2. No. 7). With the Heine-Borel Principle the mathematicians in one group were able to derive the result that a function increasing at each point of an interval is increasing on that interval (Exercises A7-2, No. 6). The concept "increasing at a point" is significant because $f'(a) > 0$ implies f is increasing at a, but not that f is increasing in any neighborhood which contains a (Compare Exercises 5-4, No. 7). Our mathematicians were stumped to find an alternative proof--perhaps one of your students can!

Exercises A7-2, Numbers 1-4 develop and apply the concept of bounded variation. This was one of the first concepts introduced by the French mathematician C. Jordan (1838-1922) in the effort to develop a theory of the integral. It remains useful for the generalization of the present concept of integral to the Riemann-Stieltjes integral and other applications.

Solutions Exercises A7-1

1. Show that the Heine-Borel Principle fails for the interval $[1,\sqrt{2})$; that is, find a cover C of $[1,\sqrt{2})$ such that no finite subset of C is a cover of $[1,\sqrt{2})$.

   If $1 \leq x < \sqrt{2}$ then choose $\delta = \frac{1}{2}(\sqrt{2} - x)$ and take the set of neighborhoods $N_\delta(x)$ as the cover of I. No finite subset of neighborhoods is a cover for suppose $\{N_{\delta_i}(x_i) ; i = 1, \ldots, n\}$ is a cover where $\delta_i = \frac{1}{2}(\sqrt{2} - x_i)$.

   Set $x = \text{Max}\{x_i\}$. Setting $\delta = \frac{1}{2}(\sqrt{2} - x)$ we see that the point $x + \delta = \frac{1}{2}(\sqrt{2} +$ [illegible]$)$ is a point of I but does not lie in any of the neighborhoods of the finite subset.

2. Prove that the Heine-Borel Principle fails for open intervals; that is, find a cover C of an open interval such that no finite subset of C is a cover.

   Let (a,b) be the open interval and cover each point x of (a,b) with the interval $I(x) = (\frac{1}{2}(a + x), \frac{1}{2}(x + b))$ which reaches from x half-way to either end-point. No finite subset of this cover will serve as a cover of (a,b). For any finite subset of the cover,

   $$\{I(x_k) : k = 1, 2, \ldots, n\}$$

   let $\xi = \text{Min}\{x_k ; k = 1, 2, \ldots, n\}$; the interval $(a, \frac{1}{2}(a + \xi))$ is not covered.

3. Complete the demonstration that the Heine-Borel Principle is equivalent to the Separation Axiom; that is, show in an ordered field that the principle implies the axiom.

   Let A and B be non-empty sets with the property $\alpha < \beta$ for each $\alpha \in A$ and $\beta \in B$. Let u and v be fixed points of A and B respectively. If $u = v$ then $s = u = v$ is a separation number because $\alpha \leq v$ for all $\alpha \in A$ and $\beta \geq u$ for all $\beta \in B$. Otherwise, we have $u < v$. In this event we shall prove that the interval $[u,v]$ contains a separation number by showing that the contrary assumption leads to a contradiction.

If $[u,v]$ does not contain a separation number then there must be at least one α in $[u,v]$ such that $\alpha \in A$ and $\alpha > u$. Otherwise we would have $\alpha \leq u$ for all $\alpha \in A$ and it would follow that u is a separation number. Similarly, there must be a $\beta \in B$ in $[u,v]$ for which $\beta < v$. We consider all neighborhoods of u of the form

$$N^{\alpha}(u) = \{x : |x - u| < \alpha - u\}$$

and neighborhoods of v of the form

$$N^{\beta}(v) = \{x : |x - v| < v - \beta\}$$

where α and β are elements of A and B in $[u,v]$ for which $\alpha > u$ and $\beta < v$.

If the set C consisting of all neighborhoods $N^{\alpha}(u)$ and $N^{\beta}(v)$ is not a cover of $[u,v]$ then there is at least one point s which is not covered. We cannot have $\alpha > s$ for any $\alpha \in A$ in $[u,v]$ since s would then be contained in $N^{\alpha}(u)$. Similarly $\beta < s$ is impossible for then s would be contained in $N^{\beta}(v)$. It follows the $\alpha \leq s$ and $\beta \geq s$ for all $\alpha \in A$, $\beta \in B$ in $[a,b]$ and consequently that s is a separation number for A and B, a contradiction to our assumption.

On the other hand, if C is a cover of $[u,v]$ then there must be a finite subset of C which covers $[u,v]$. Among its members the finite subset must contain at least one of each; a neighborhood $N^{\alpha}(u)$ of u and a neighborhood $N^{\beta}(v)$ of v, since no $N^{\alpha}(u)$ covers v and no $N^{\beta}(v)$ covers u. Let $N^{\overline{\alpha}}(u)$ denote the largest neighborhood of u in the finite cover, and $N^{\overline{\beta}}(v)$ the largest neighborhood of v. The intervals $N^{\overline{\alpha}}(u)$ and $N^{\overline{\beta}}(v)$ must be a cover because $N^{\overline{\alpha}}(u)$ contains all neighborhoods $N^{\alpha}(u)$ and $N^{\overline{\beta}}(v)$ contains all neighborhoods $N^{\beta}(v)$ of the finite cover. Since $\overline{\alpha} \in A$ and $\overline{\beta} \in B$ we have $\overline{\alpha} \leq \overline{\beta}$ and therefore $\overline{\alpha}$ and $\overline{\beta}$ are not covered. (It follows that $\overline{\alpha}$ and $\overline{\beta}$ are separation points but the contradiction is already established.)

In short, whether or not C is a cover of $[u,v]$, under the assumption that no separation point exists, we obtain a contradiction.

4. State and prove the converse of the Heine-Borel Principle.

If every cover C of the interval I contains a finite cover, then I is a closed interval.

Proof. If I were open (See No. 2) or half-open (See No. 1) then there would exist a cover C containing no finite cover in contradiction to the hypothesis.

5. Since the Separation Axiom fails for the field of rational numbers, so also must the Heine-Borel Principle. State the would-be Heine-Borel Principle for rational numbers and show by example that it is not valid.

Let $I = [a,b]$ where a and b are rational numbers. A set C of open intervals with rational endpoints is called a cover of I if every rational point in I is covered by one of the intervals in C. If the Heine-Borel Principle were valid in the field of rational numbers then every cover C would contain a finite subset of intervals which form a cover.

Consider the interval $[0,1]$. Cover 0 and 1 with neighborhoods of radius $\frac{1}{6}$. For any rational number given in the form $\frac{p}{q}$ in lowest terms with $1 \le p \le q - 1$ take a neighborhood of radius $\frac{1}{2(q-1)}\left(\frac{1}{3}\right)^q$. The sum S_q of the lengths of all neighborhoods corresponding to such irreducible fractions with denominator q (for $q > 1$) then satisfies

$$S_q \le 2(q-1)\left[\frac{1}{2(q-1)}\left(\frac{1}{3}\right)^q\right] \le \left(\frac{1}{3}\right)^q .$$

Summing over all q, we obtain

$$\sum_{q=1}^{\infty} S_q = \lim_{n \to \infty} \sum_{q=1}^{n} S_q \le \frac{2}{3} \sum_{q=2}^{\infty} \left(\frac{1}{3}\right)^q$$

where the last sum is that of an infinite geometrical progression with common ratio $\frac{1}{3}$. Consequently, $\sum_{q=1}^{\infty} S_q \le \frac{5}{6} < 1$. Now if a finite subset of these neighborhoods could be chosen to cover every rational point, then their total length would be less than 1; consequently, some interval must be left uncovered. But every interval contains rational points; consequently, no interval can be left uncovered. The assumption of a finite cover has yielded a contradiction.

Solutions Exercises A7-2

1. A function is said to be of bounded variation on $[a,b]$ if there exists a bound M such that for all partitions $\sigma = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a,b]$.

$$\lambda(\sigma) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq M .$$

(a) Prove if the function f is monotone on $[a,b]$ then f is of bounded variation on $[a,b]$.

If f is weakly increasing then

$$f(x_k) \geq f(x_{k-1})$$

and

$$|f(x_k) - f(x_{k-1})| = f(x_k) - f(x_{k-1}) .$$

If f is weakly decreasing then

$$|f(x_k) - f(x_{k-1})| = f(x_{k-1}) - f(x_k) .$$

In either case

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| = |f(b) - f(a)|$$

independently of the choice of subdivision.

(b) Prove if f is a function of bounded variation then f can be represented as a sum $f = g + h$ where g is weakly increasing and h is weakly decreasing. Prove conversely, if f can be represented as such a sum of monotone functions, then f is a function of bounded variation.

If $f = g + h$ where g is weakly increasing and h is weakly decreasing, then

$$\begin{aligned}
\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| &= \sum_{k=1}^{n} |g(x_k) - g(x_{k-1}) + h(x_k) - h(x_{k-1})| \\
&\leq \sum_{k=1}^{n} ([g(x_k) - g(x_{k-1})] + [h(x_{k-1}) - h(x_k)]) \\
&\leq \sum_{k=1}^{n} [g(x_k) - g(x_{k-1})] - \sum_{k=1}^{n} [h(x_k) - h(x_{k-1})] .
\end{aligned}$$

The sums in the last inequality telescope and we obtain

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq [g(b) - g(a)] - [h(b) - h(a)]$$

which yields the desired bound.

In order to prove the primary proposition it is natural, by analogy with our proof that a piecewise monotone function can be represented in the desired form (Section 7-1), to attempt to decompose the sum $\lambda(\sigma)$ into two parts: a sum $\mu(\sigma)$ of all the terms for which $f(x_k) - f(x_{k-1}) > 0$ and a sum $\nu(\sigma)$ of all terms for which $f(x_k) - f(x_{k-1}) < 0$. The sums $\mu(\sigma)$ and $\nu(\sigma)$ can be represented explicitly in the form

$$\mu(\sigma) = \sum_{k=1}^{n} a_k \;, \quad \nu(\sigma) = \sum_{k=1}^{n} b_k \;,$$

where

$$a_k = \frac{1}{2}\{|f(x_k) - f(x_{k-1})| + f(x_k) - f(x_{k-1})\} \;,$$

$$b_k = \frac{1}{2}\{|f(x_k) - f(x_{k-1})| - (f(x_k) - f(x_{k-1}))\} \;.$$

Not only do we have

(1) $$\mu(\sigma) + \nu(\sigma) = \lambda(\sigma)$$

but also

(2) $$\mu(\sigma) - \nu(\sigma) = \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})] = f(b) - f(a) \;.$$

Equations (1) and (2) offer a suggestion for simplifying our approach. Since

(3) $$\mu(\sigma) = \frac{1}{2}\{\lambda(\sigma) + [f(b) - f(a)]\}$$

and

(4) $$\nu(\sigma) = \frac{1}{2}\{\lambda(\sigma) - [f(b) - f(a)]\}$$

we see that the properties of $\mu(\sigma)$ and $\nu(\sigma)$ can be derived from those of $\lambda(\sigma)$ alone. (It would be possible to work with $\mu(\sigma)$ and $\nu(\sigma)$ without the use of this device but this increases the number of details to be settled in the course of the proof.)

Proof. In general, for each $x > a$ in $[a,b]$ and each parition σ_x of $[a,x]$ we introduce a sum $\lambda(\sigma_x)$ which plays the same role for $[a,x]$ as $\lambda(\sigma)$ does for $[a,b]$:

$$\mu(\sigma_x) = \sum_{k=1}^{p} |f(u_k) - f(u_{k-1})|$$

where

$$\sigma_x = \{u_0, u_1, u_2, \ldots, u_p\} ,$$

$u_0 = a$, and $u_p = x$.

Since σ_x can be extended to a partition $\bar{\sigma}_x$ of $[a,b]$ by the addition of the point b , and

(5) $$\lambda(\bar{\sigma}_x) = \lambda(\sigma_x) + |f(b) - f(x)|$$

we see that for all x and σ_x

$$\lambda(\sigma_x) \leq M .$$

The least upper bound (See Appendix A1-5) of $\lambda(\sigma_x)$ is denoted by $T(x)$ and called the total variation of f on $[a,x]$. For convenience, we define $T(a) = 0$.

The remainder of the proof consists of showing that the functions $T + f$ and $T - f$ are weakly increasing (compare (3) and (4) above). Setting $g(x) = \frac{1}{2}[T(x) + f(x)]$ and $h(x) = \frac{1}{2}[T(x) - f(x)]$ we shall then have the result of the theorem. Consider the values of $T + f$ for points u , v of $[a,b]$ when $u < v$. Since $T(u)$ is the least upper bound of $\lambda(\sigma_x)$ we know from Corollary 1 to the Least Upper Bound Principle that for each positive ϵ there exists a partition σ_u of $[a,u]$ for which

(6) $$\lambda(\sigma_u) > T(u) - \epsilon .$$

Now, we extend σ_u to a partition $\bar{\sigma}_u$ of $[a,v]$ by the addition of the point v , and obtain as in (5),

$$\lambda(\bar{\sigma}_u) = \lambda(\sigma_u) + |f(v) - f(v)| .$$

Since $T(v) \geq \lambda(\bar{\sigma}_u)$, it follows from (6) that

(7) $$T(v) > T(u) + |f(v) - f(u)| - \epsilon .$$

From (7), we have

$$\begin{aligned} T(v) + f(v) &> T(u) + f(v) + |f(v) - f(u)| - \varepsilon \\ &\geq T(u) + f(v) + [f(u) - f(v)] - \epsilon \\ &\geq T(u) + f(u) - \epsilon \ , \end{aligned}$$

and, since this inequality holds for all positive values ϵ ,

$$T(v) + f(v) \geq T(u) + f(u) \ .$$

We have proved that $T + f$ is weakly increasing. It follows that $T - f$ is weakly increasing, for each sum $\lambda(\sigma_x)$ is the same for $-f$ as it is for f , and therefore T is also the total variation of $-f$. The theorem then follows from the observation of the preceding paragraph.

2. The preceding result enables us to prove that not all integrable functions are linear combinations of monotone functions. Consider the function defined by

$$f(x) = \begin{cases} x \cos \frac{\pi}{x} \ , & x \neq 0 \\ 0 \ , & x = 0 \ . \end{cases}$$

Prove that f is continuous, therefore integrable on $[0,1]$.
Prove also that f is not of bounded variation on $[0,1]$.

The function f is continuous for $x \neq 0$ because it is differentiable. For $x = 0$, note that

$$\begin{aligned} |f(x) - f(0)| &= |x| \cdot \left|\cos \frac{\pi}{x}\right| \\ &\leq |x| \\ &< \epsilon \end{aligned}$$

whenever $|x| < \epsilon$.

To show that f is not of bounded variation utilize the values

$$f(\frac{1}{n}) = \frac{1}{n} \cos \pi n = \frac{(-1)^n}{n} \ .$$

For the partition of $[0,1]$, where

$$\sigma = \{0 , \frac{1}{2^p} , \frac{1}{2^p - 1} , \ldots , \frac{1}{3} , \frac{1}{2} , 1\}$$

we have, on reversing the order of terms,

$$\begin{aligned} \lambda(\sigma) = (1 + \tfrac{1}{2}) + (\tfrac{1}{2} + \tfrac{1}{3}) + (\tfrac{1}{3} + \tfrac{1}{4}) \\ + \ldots + (\frac{1}{2^p - 1} + \frac{1}{2^p}) + (\frac{1}{2^p} + 0) \ . \end{aligned}$$

Regroup the terms as follows,

$$\lambda(\sigma) = 1 + 2(\tfrac{1}{2}) + 2(\tfrac{1}{3} + \tfrac{1}{4}) + 2(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8})$$
$$+ \ldots + 2(\frac{1}{2^{p-1} + 1} + \ldots + \frac{1}{2^p}) .$$

Set

$$a_k = \frac{1}{2^{k-1} + 1} + \frac{1}{2^{k-1} + 2} + \ldots + \frac{1}{2^k} .$$

and observe that the smallest term in a_k is the last. Since there are $2^k - 2^{k-1} = 2^{k-1}(2 - 1) = 2^{k-1}$ terms in a_k we have

$$a_k \geq \frac{2^{k-1}}{2^k} \geq \frac{1}{2} .$$

However,

$$\lambda(\sigma) = 1 + 2 \sum_{k=1}^{p} a_k$$
$$\geq 1 + 2 \sum_{k=1}^{p} \frac{1}{2}$$
$$\geq 1 + p .$$

Since p may be any positive integer whatever, it follows that there can be no upper bound on the sums $\lambda(\sigma)$.

3. Prove if f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b) and if the derivative f' is bounded on (a,b) , then f is of bounded variation on $[a,b]$.

Suppose $|f'(x)| \leq M$, on $[a,b]$. Applying the Law of the Mean, we obtain

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| = \sum_{k=1}^{n} |f'(u_k)|(x_k - x_{k-1}) ,$$
$$\leq \sum_{k=1}^{n} M(x_k - x_{k-1})$$
$$\leq M(b - a)$$

where u_k is some value in the open interval (x_{k-1},x_k) .

4. Prove that if the function f is of bounded variation on $[a,b]$ and if $a \leq c < d \leq b$ then f is of bounded variation on $[c,d]$.

Extend any partition σ of $[c,d]$ to a partition $\bar{\sigma}$ of $[a,b]$ by adjoining the points a and b; then $\lambda(\sigma) \leq \lambda(\bar{\sigma}) \leq M$.

5. The concept of length of a curve, like that of area, is not defined in general by the methods of elementary geometry. By analogy with the concept of integral, it is natural to attempt to express the length of a curve as a limit of the lengths of polygonal approximations. Let a curve be given as the graph of a given function f on $[a,b]$. Given a partition $\sigma = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ we construct an inscribed polygon to the graph of f by joining the successive points (x_k, y_k) where $y_k = f(x_k)$, for $k = 0, 1, 2, \ldots, n$ (Figure A7-2). Let $\mathcal{L}$ denote the length of the graph and $L(\sigma)$ the length of the inscribed polygon. We have

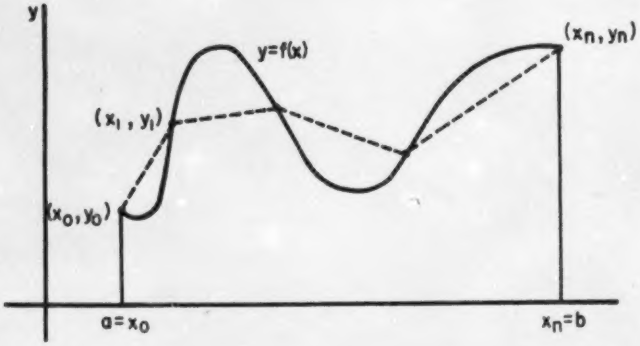


Figure A7-2

$$L(\sigma) = \sum_{k=1}^{n} \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}$$

where the general term in the sum is the length of the straight-line segment joining the points (x_{k-1}, y_{k-1}) and (x_k, y_k). Intuitively, the straight path is the shortest path between the two points, so that the path along the graph of f is never shorter than the segment joining the two points. We must then have $L(\sigma) \leq \mathcal{L}$ for all partitions σ. We can only estimate $\mathcal{L}$ from below by using inscribed polygons. Furthermore, there is no obvious general way of estimating $\mathcal{L}$ from above. For this reason we define $\mathcal{L}$ as the least upper bound of the lengths $L(\sigma)$ of inscribed polygons, provided such a bound exists. If the length $\mathcal{L}$ exists we say that f is rectifiable over $[a,b]$. Prove that a necessary and sufficient condition for f to be rectifiable over $[a,b]$ is that f be of bounded variation on $[a,b]$.

If f is rectifiable then for the sums $L(\sigma)$ we have an upper bound:

$$L(\sigma) = \sum_{k=1}^{n} \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2} \leq \mathcal{L} .$$

Since $|y_k - y_{k-1}| = \sqrt{(y_k - y_{k-1})^2} \leq \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}$

we have

$$\sum_{k=1}^{n} |y_k - y_{k-1}| = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq \mathcal{L} ,$$

i.e., f is a function of bounded variation.

If f is a function of bounded variation, then for the sums $\lambda(\sigma)$ we have an upper bound:

$$\lambda(\sigma) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq M .$$

Since

$$\sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2} \leq |x_k - x_{k-1}| + |y_k - y_{k-1}|$$

we have

$$\begin{aligned} L(\sigma) &= \sum_{k=1}^{n} \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2} \\ &\leq \sum_{k=1}^{n} (x_k - x_{k-1}) + \sum_{k=1}^{n} |y_k - y_{k-1}| \\ &\leq b - a + \lambda(\sigma) \\ &\leq b - a + M . \end{aligned}$$

Thus $\{L(\sigma)\}$ is bounded above and has a least upper bound $\mathcal{L}$.

6. Let α be a point of the domain of f for which every deleted neighborhood contains other points of the domain. The function f is said to be increasing at the point α if there exists some neighborhood wherein

(i) $x < \alpha \Rightarrow f(x) < f(\alpha)$

and

(ii) $x > \alpha \Rightarrow f(x) > f(\alpha)$

for those points x in the domain of f. Show that if f is increasing at every point of the interval (a,b) then f is increasing on (a,b).

Let u and v be any points of (a,b) with $u < v$. For each point α of $[u,v]$ there is a neighborhood of α wherein (i) and (ii) are satisfied. The set of such neighborhoods forms a cover of $[u,v]$. Let C be any finite cover contained in the set of neighborhoods. If any of the neighborhoods from the set C is covered by the others it can be removed from C and the set will remain a cover. We may therefore assume that no member of C is contained in the union of the others. Let the centers of the neighborhoods from C be given in increasing order,

$$\alpha_1 < \alpha_2 < \dots < \alpha_k ,$$

and let $N_1, N_2, \dots, N_k$ be the corresponding neighborhoods. Each pair of consecutive neighborhoods overlap, $N_i \cap N_{i+1}$ is non-empty. For if the two neighborhoods did not have a point in common, there would be some point x in (α_i, α_{i+1}) not covered by either N_i or N_{i+1}. Some x is covered by some neighborhood, say N_j with $j < i$, it follows that α_i and hence all of N_i is covered by N_j, in contradiction to the assumption that no member of C is covered by the others. From this argument, it follows that there is a point β_i in (α_i, α_{i+1}) covered by both N_i and N_{i+1}. Furthermore, in the same way, it follows that $u \in N_1$ and $v \in N_2$. Consequently, from Properties (i) and (ii)

$$f(u) < f(\alpha_1) < f(\beta_1) < f(\alpha_2) < f(\beta_2) \dots < f(\alpha_k) < f(v) ;$$

thus the increasing character of f is proved.

7. By the Heine-Borel Principle prove that if f is continuous on $[a,b]$ then f is bounded on $[a,b]$, (Theorem A4-1).

For each α in $[a,b]$ there exists a neighborhood $|x - \alpha| < \delta(\alpha)$ wherein $|f(x) - f(\alpha)| < 1$. The set of such neighborhoods forms a cover of $[a,b]$. Let C be any finite cover made up of such neighborhoods. Let the centers of the neighborhoods from C be $\alpha_1, \alpha_2, \dots, \alpha_k$. For each x in $[a,b]$ there is some α_i for which $|x - \alpha_i| < \delta(\alpha_i)$, hence $|f(x) - f(\alpha_i)| < 1$, and $|f(x)| < |f(\alpha_i)| + 1$. Consequently, for all $x \in [a,b]$,

$$|f(x)| < \max \{|f(\alpha_i)| + 1\} ;$$

i.e., $f(x)$ is bounded.

Teacher's Commentary

Appendix 8

Solutions Exercises A8

1. (a) **Employing the methods of the text, extend the definitions of sin : $x \longrightarrow \phi(x)$ and cos : $x \longrightarrow \psi(x)$ given by (4), (5), and (6) to all negative values of x.**

Proceed as in the text, but take $x_0 = -2^k\xi$ where the discussion in the text uses $x_0 = 2^k\xi$.

(b) **Use the fact that the functions ϕ and ψ have been defined for $x \geq -\xi$, where $0 < \xi < 1$, to extend their domain to include the set of all negative numbers.**

The function $\hat{\phi} : x \longrightarrow -\phi(-x)$ is defined for $x \leq \xi$ and satisfies (1) and (2). By the uniqueness theorem $\hat{\phi} = \phi$ on $[-\xi, \xi]$; hence,

$$\sin : x \longrightarrow \begin{cases} \phi(x) \ , & x > \xi \\ \phi(x) = \hat{\phi}(x) \ , & -\xi \leq x \leq \xi \\ \hat{\phi}(x) \ , & x < -\xi \end{cases}$$

is the unique extension of ϕ to the domain of all real numbers. Finally, define $\hat{\psi} = \hat{\phi}'$ and cos , accordingly.

2. **(This exercise employs the symmetries of ϕ and ψ from Exercises 8-5, No. 11, and the addition theorems, Exercises 8-5, Nos. 12 and 13.) Prove:**

(a) $\phi(\frac{\pi}{2} - x) = \psi(x)$.

$$\begin{aligned} \phi(\tfrac{\pi}{2} - x) &= \phi(\tfrac{\pi}{2})\psi(-x) + \phi(-x)\psi(\tfrac{\pi}{2}) \\ &= \psi(-x) \qquad (\text{from } \phi(\tfrac{\pi}{2}) = 1 \ , \ \psi(\tfrac{\pi}{2}) = 0) \\ &= \psi(x) \ . \end{aligned}$$

(b) $\psi(\frac{\pi}{2} - x) = \phi(x)$.

$$\psi(\tfrac{\pi}{2} - x) = \psi(\tfrac{\pi}{2})\psi(-x) - \phi(\tfrac{\pi}{2})\phi(-x) = \phi(x) \ .$$

(c) $\phi(\pi - x) = \phi(x)$.

$$\phi(\pi - x) = \phi(\pi)\psi(-x) + \phi(-x)\psi(\pi)$$
$$= -\phi(-x) = \phi(x) .$$

(d) $\psi(\pi - x) = -\psi(x)$.

$$\psi(\pi - x) = \psi(\pi)\psi(-x) - \phi(\pi)\phi(-x)$$
$$= -\psi(-x) = -\psi(x) .$$

3. Show

(a) $\phi(\frac{\pi}{6}) = \frac{1}{2}$.

From Number 1(b),

$$\phi(\tfrac{\pi}{6}) = \psi(\tfrac{\pi}{2} - \tfrac{\pi}{6}) = \psi(\tfrac{\pi}{6} + \tfrac{\pi}{6}) = \psi^2(\tfrac{\pi}{6}) - \phi^2(\tfrac{\pi}{6})$$
$$= 1 - 2\phi^2(\tfrac{\pi}{6}) .$$

The quadratic equation

$$2\phi^2(\tfrac{\pi}{6}) + \phi(\tfrac{\pi}{6}) - 1 = [2\phi(\tfrac{\pi}{6}) - 1][\phi(\tfrac{\pi}{6}) + 1] = 0$$

has the roots $\frac{1}{2}$ and 0 . But, since $0 < \frac{\pi}{6} < \frac{\pi}{4}$, $\phi(\frac{\pi}{6}) > 0$. Consequently $\phi(\frac{\pi}{6}) = \frac{1}{2}$.

(b) $\psi(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$.

From (a), $\psi^2(\frac{\pi}{6}) = 1 - \phi^2(\frac{\pi}{6}) = \frac{3}{4}$. In the text it has been shown that $\psi(x) > 0$ on $[0, \frac{\pi}{4}]$. Consequently $\psi(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$.

(c) $\phi(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$.

From Number 2(a) and Number 3(b),

$$\phi(\tfrac{\pi}{3}) = \psi(\tfrac{\pi}{2} - \tfrac{\pi}{3}) = \psi(\tfrac{\pi}{6}) = \tfrac{\sqrt{3}}{2} .$$

(d) $\psi(\frac{\pi}{3}) = \frac{1}{2}$.

From Number 2(b) and Number 3(a),

$$\psi(\tfrac{\pi}{3}) = \phi(\tfrac{\pi}{6}) = \tfrac{1}{2} .$$

4. Show:

(a) ϕ has no positive period less than 2π .

Let $p > 0$ be a period of ϕ . Then $\phi(p) = \phi(0 + p) = \phi(0) = 0$. Since ϕ is increasing on $[0, \frac{\pi}{4}]$ we know that $\phi(x) > 0$ for $0 < x \leq \frac{\pi}{4}$. Consequently $\psi(x) = \phi(\frac{\pi}{2} - x) > 0$ for $\frac{\pi}{4} \leq x < \frac{\pi}{2}$; i.e., $\phi'(x) = \psi(x)$ is positive on the interval. It follows that $\phi(x)$ is increasing on $[\frac{\pi}{4}, \frac{\pi}{2}]$ also. Hence, $\phi(x) > 0$ for $0 < x \leq \frac{\pi}{2}$. From $\phi(\pi - x) = \phi(x)$, then,

$$\phi(x) > 0 \text{ , for } x \in (0,\pi) .$$

It follows that π is the smallest possible period. Furthermore no number between π and 2π can be a period, for if λ were such a period, then $2\pi - \lambda$ is a period which is a contradiction since $0 < 2\pi - \lambda < \pi$. Moreover, π is not a period, since

$$\begin{aligned}\phi(\pi + x) &= \phi(\pi)\psi(x) + \phi(x)\psi(\pi)\\ &= -\phi(x) ;\end{aligned}$$

but $-\phi(x) \neq \phi(x)$ when $\phi(x) \neq 0$. We conclude that 2π is the smallest positive period.

(b) the function $\tau : x \longrightarrow \frac{\phi(x)}{\psi(x)}$ is periodic with period π .

In the solution of (a) it was shown that $\phi(x + \pi) = -\phi(x)$. Similarly,

$$\psi(x + \pi) = \psi(x)\psi(\pi) - \phi(x)\phi(\pi) = -\psi(x) .$$

Consequently,

$$\tau(x + \pi) = \frac{\phi(x + \pi)}{\psi(x + \pi)} = \frac{\phi(x)}{\psi(x)} = \tau(x) .$$

5. Derive the formulas $\phi(z + 2\pi) = \phi(z)$ and $\psi(z + 2\pi) = \psi(z)$

   (a) from (13).

   (b) directly from (11) and the addition theorems.

(a) If $f(x) = f(x - 2\pi)$, then for $x = z + 2\pi$

$$f(z + 2\pi) = f(z)$$

(b) Since $\phi(2\pi) = 0$, $\psi(2\pi) = 1$,

$$\phi(x + 2\pi) = \phi(x)\psi(2\pi) + \psi(x)\phi(2\pi) = \phi(x) ,$$

$$\psi(x + 2\pi) = \psi(x)\psi(2\pi) - \phi(x)\phi(2\pi) = \psi(x) .$$

6. Prove that if t is the arclength of the curve $y = \sqrt{1 - x^2}$ between $x = 0$ and $x = a$ then $\phi(t) = a$ and $\psi(t) = \sqrt{1 - a^2}$.

In this exercise we verify the connection of the circular functions with geometry. Thus the analytic definition of the circular functions validates their intuitive description.

From Section 6-3(iv),

$$t = \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx .$$

But,

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^2}{1 - x^2}} = \frac{1}{\sqrt{1 - x^2}} .$$

Thus,

$$t = \int_0^a \frac{dx}{\sqrt{1 - x^2}} ,$$

hence, from Equation (4) and (5), $a = \phi(t)$, and, from $\psi(t) = \sqrt{1 - \phi(t)^2}$ in a neighborhood of $t = 0$, $\psi(t) = \sqrt{1 - a^2}$.

Teacher's Commentary

Appendix 10

CONVERGENCE OF IMPROPER INTEGRALS

The proof of Theorem 10-6a is put in the appendix because it involves a form of the completeness axiom for the reals (the least upper bound principle is used in Lemma A10a).

Solutions Exercises A10

1. Complete the proof of Lemma A10a.

It is left to prove the case for the right-sided limit at a when ϕ is weakly decreasing and both cases for the left-sided limit at b.

If ϕ is weakly decreasing in a right neighborhood of a consider $\lim_{x \to a^+} -\phi(x)$ since $-\phi$ is weakly increasing and satisfies the other criteria used for the proof in the text. Then

$$\lim_{x \to a^+} \phi(x) = -\lim_{x \to a^+} -\phi(x) .$$

To prove the result for a left-sided limit at b, consider

$$\psi(x) = \phi(-x)$$

and observe that

$$\lim_{x \to (-b)^+} \psi(x) = \lim_{x \to b^-} \phi(x) ;$$

the case for a left-sided limit is thereby reduced to the case for a right-sided limit.

2. Show if ψ is weakly increasing on (ξ, b) and has a left-sided limit at b, $K = \lim_{\beta \to b^-} \psi(\beta)$, then ψ is bounded on (ξ, b).

Since $K = \lim_{\beta \to b^-} \psi(\beta)$ there exists a deleted left-sided neighborhood $(b - \delta, b)$ where $|\psi(\beta) - K| < \epsilon$. Hence $\psi(b) < K + \epsilon$ for $\beta \in (b - \delta, b)$. For any points β, γ satisfying $\xi < \gamma \leq b - \delta < \beta < b$

from the monotone property

$$\psi(\gamma) \leq \psi(\beta) < K + \epsilon \ .$$

It follows that ψ is bounded on (ξ, b) .

3. Complete the proof of Theorem A10a for $\int_a^\eta f(x)dx$.

In the notation of the text we have for $\alpha \in (a,\eta)$,
$\phi(\alpha) = \int_\alpha^\eta g(x)dx = \int_\xi^\beta g(-t)dt = \psi(\beta)$ where $\beta = -\alpha$. Now
$\lim_{\alpha \to a^+} \phi(\alpha)$ exists. But

$$\lim_{\alpha \to a^+} \phi(\alpha) = \lim_{\alpha \to a^+} \psi(-\alpha) = \lim_{\beta \to b^-} \psi(\beta) \ .$$

Furthermore

$$|f(-t)| = |f(x)| \leq g(x) = g(-t) \ .$$

Thus $f^*(t) = f(-t)$ and $g^*(t) = g(-t)$ satisfy the conditions of the first part of the theorem, from which the desired result follows.

4. Show that the condition in Theorem 10-6a, that f be integrable over every closed subinterval of (a,b) , cannot be omitted from the convergence criterion. More precisely, show that if $|f(x)| < g(x)$ and $\int_a^b g(x)dx$ exists, then $\int_a^b |f(x)|dx$ exists but $\int_a^b f(x)dx$ need not exist.

Consider the solution of Exercises 6-4, Number 22(b).

5. By estimating the absolute difference between the areas of successive arches of the curve $y = \frac{\sin x}{x}$, i.e.,

$$A_n = \int_{2(n-1)\pi}^{2n\pi} \frac{\sin x}{x} dx \ , \qquad (n = 1 \ , 2 \ , 3 \ , \ldots)$$

show that

$$I = \int_0^\infty \frac{\sin x}{x} dx$$

converges.

Write

$$A_n = \left[\int_{(2n-2)\pi}^{(2n-1)\pi} \frac{\sin x}{x}\, dx + \int_{(2n-1)\pi}^{2n\pi} \frac{\sin x}{x}\, dx \right] .$$

Use the substitution $x = u + (2n - 2)\pi$ for the first integral, and $x = v + (2n - 1)\pi$ for the second, to obtain

$$\begin{aligned} A_n &= \int_0^\pi \frac{\sin u}{u + (2n - 2)\pi}\, du - \int_0^\pi \frac{\sin u}{v + (2n - 1)\pi}\, dv \\ &= \int_0^\pi \sin x \left[\frac{1}{x + (2n - 2)\pi} - \frac{1}{x + (2n - 1)\pi} \right] dx \\ &= \int_0^\pi \frac{\pi \sin x}{[x + (2n - 2)\pi][x + (2n - 1)\pi]}\, dx . \end{aligned}$$

From the inequalities $0 \le \sin x \le 1$ and $[x + (2n - 2)\pi][x + (2n - 1)\pi] > 4(n - 1)^2$ on $[0,\pi]$, it follows that

$$0 < A_n < \frac{1}{4(n - 1)^2} , \qquad (n = 2 , 3 , 4 , \ldots) .$$

This result enables us to bound

$$I_n = \int_0^{2n\pi} \frac{\sin x}{x}\, dx = \sum_{k=1}^{n} A_k .$$

Now, for $m > 1$,

$$\frac{1}{m^2} = \int_{m-1}^{m} \frac{1}{m^2}\, dt \le \int_{m-1}^{m} \frac{1}{t^2}\, dt .$$

Thus,

$$\begin{aligned} I_n &= A_1 + A_2 + A_3 + \ldots + A_n \\ &< A_1 + \frac{1}{4} + \frac{1}{4 \cdot 2^2} + \ldots + \frac{1}{4(n - 1)^2} \\ &\le A_1 + \frac{1}{4} + \frac{1}{4} \left[\int_1^2 \frac{1}{t^2}\, dt + \ldots + \int_{n-2}^{n-1} \frac{1}{t^2}\, dt \right] \\ &\le A_1 + \frac{1}{4} + \frac{1}{4} \int_1^{n-1} \frac{1}{t^2}\, dt \\ &\le A_1 + \frac{1}{4} + \frac{1}{4} - \frac{1}{4}\left(\frac{1}{n - 1}\right) \\ &< A_1 + \frac{1}{2} . \end{aligned}$$

Thus I_n is bounded and monotone, hence by Lemma A10-a, $\lim_{n\to\infty} I_n$ exists.

To prove $\int_0^\infty \frac{\sin x}{x}\,dx$ converges, consider

$$\phi(z) = \int_0^z \frac{\sin x}{x}\,dx .$$

Set $n = \left[\!\left[\frac{z}{2\pi}\right]\!\right]$. Since $0 < z - 2n\pi < 2\pi$, we have

$$\begin{aligned} |\phi(z) - I_n| &= \left| \int_{2n\pi}^{z} \frac{\sin x}{x}\,dx \right| \\ &\le \int_{2n\pi}^{z} \left|\frac{\sin x}{x}\right| dx \\ &\le 2\pi \cdot \frac{1}{2n\pi} \\ &\le \frac{1}{n} . \end{aligned}$$

6. Prove that $\int_0^1 \frac{1}{x} \sin \frac{1}{x}\,dx$ converges.

Set $x = \frac{1}{u}$; then

$$I = \int_1^\infty \frac{\sin u}{u}\,du$$

which converges by the proof of Number 5. Thus $\int_0^1 \frac{1}{x} \sin \frac{1}{x}\,dx$ converges by Exercises 10-6b, Number 2(d) and use of the inverse substitution $u = \frac{1}{x}$.